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MODEL OF THE FIELD LINE RANDOM WALK EVOLUTION AND APPROACH TO ASYMPTOTIC DIFFUSION IN MAGNETIC TURBULENCE

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ABSTRACT

The turbulent random walk of magnetic field lines plays an important role in the transport of plasmas and energetic particles in a wide variety of astrophysical situations, but most theoretical work has concentrated on determination of the asymptotic field line diffusion coefficient. Here we consider the evolution with distance of the field line random walk using a general ordinary differential equation (ODE), which for most cases of interest in astrophysics describes a transition from free streaming to asymptotic diffusion. By challenging theories of asymptotic diffusion to also describe the evolution, one gains insight on how accurately they describe the random walk process. Previous theoretical work has effectively involved closure of the ODE, often by assuming Corrsin's hypothesis and a Gaussian displacement distribution. Approaches that use quasilinear theory and prescribe the mean squared displacement $\langle \Delta x^2 \rangle$ according to free streaming (random ballistic decorrelation, RBD) or asymptotic diffusion (diffusive decorrelation, DD) can match computer simulation results, but only over specific parameter ranges, with no obvious "marker" of the range of validity. Here we make use of a unified description in which the ODE determines $\langle \Delta x^2 \rangle$ self-consistently, providing a natural transition between the assumptions of RBD and DD. We find that the minimum kurtosis of the displacement distribution provides a good indicator of whether the self-consistent ODE is applicable, i.e., inaccuracy of the self-consistent ODE is associated with non-Gaussian displacement distributions.

Key words: diffusion – magnetic fields – turbulence

Online-only material: color figures

1. INTRODUCTION

Magnetic fields are ubiquitous in tenuous astrophysical plasmas, and because of their large distance scales, flows of such plasmas often exhibit magnetic turbulence. A magnetic field line guides the motion of charged particles, be they components of the bulk plasma or high energy particles, so a wide variety of transport phenomena depend on the field line trajectories. Therefore, numerous studies have examined the nature of the field line random walk (FLRW; e.g., Barghouty & Jokipii 1996; Zimbardo & Veltri 1995), especially with regard to asymptotic diffusion coefficients after long distances (for reviews, see Isichenko 1991a, 1991b; Shalchi 2009).

In the present work we address the evolution of the FLRW as a function of distance z along a large-scale magnetic field, an issue that has received relatively less attention. For most cases of astrophysical interest there is a transition from free-streaming (ballistic) trajectories at low z to approaching asymptotic diffusion at high z, although when the FLRW is dominated by nearly two-dimensional (2D) fluctuations, which mainly vary in the x- and y-directions, computer simulations have exhibited subdiffusion over certain distance scales (Ruffolo et al. 2008; Ghilea et al. 2011). Similar behaviors have been reported from simulations of the mathematically analogous situation of trajectories of tracers in 2D hydrodynamic turbulence (Ottaviani 1992).

In addition to its intrinsic interest, in the present work we show that the evolution of the FLRW can shed light on how well theories of asymptotic diffusion capture the correct behavior of the field line displacement distribution. The evolution of the mean squared displacement of turbulent field lines or analogous systems has been described by a second-order ordinary differential equation (ODE), and we describe how the FLRW theories in the literature correspond to various types of closure of the ODE, including self-closure that retains the form of an ODE (Saffman 1962; Taylor & McNamara 1971; Lerche 1973; Wang et al. 1995; Shalchi & Kourakis 2007), and integrable closures such as diffusive decorrelation (DD; Salu & Montgomery 1977; Matthaeus et al. 1995) and random ballistic decorrelation (RBD; Ghilea et al. 2011). The latter authors found that theories based on either DD or RBD could provide an accurate match to direct computer simulation results for the asymptotic diffusion coefficient, depending on the range of physical parameters. Indeed, the field lines might be expected to undergo a transition from RBD (free streaming) to DD (asymptotic diffusion) behavior as a function of distance, z. We find that in some cases the ODE matches the simulation results for the FLRW evolution quite well, and better than RBD or DD theories. In other cases, where such an ODE yields a running diffusion coefficient that deviates substantially (by $\gtrsim$30%) from simulation results, we typically find a minimum kurtosis K below 2.8. Therefore, although it is more complicated than most previous descriptions of asymptotic field line diffusion, this ODE provides a unified description of the FLRW evolution, and the kurtosis of the displacement distribution as determined by computer simulations provides an indicator of its accuracy.

2. ORDINARY DIFFERENTIAL EQUATION FOR FIELD LINE RANDOM WALK

2.1. General Form

The problem of the FLRW is to statistically summarize the results of tracing magnetic field lines through space. We consider a magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ for a constant large-scale field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and a fluctuating field $\mathbf{b}$ of mean zero. In the present work, we consider the case of transverse fluctuations $\mathbf{b} \perp \hat{\mathbf{z}}$, in which case the field line never backtracks as a function of z and is described by $x(z)$ and $y(z)$ as given by

$$\frac{dx}{dz} = \frac{b_x(x,y,z)}{B_0} \qquad \frac{dy}{dz} = \frac{b_y(x,y,z)}{B_0}. \tag{1}$$

(This could easily be generalized to allow non-transverse fluctuations by using three equations for $x(\tau)$, $y(\tau)$, and $z(\tau)$, where τ parameterizes the location along the field line.) Considering field line displacements $\Delta x \equiv x(z) - x(0)$ and (equivalently) Δy, the problem of the FLRW is to then find the mean squared displacements (variances)

$$V_x(z) \equiv \langle(\Delta x)^2\rangle \qquad V_y(z) \equiv \langle(\Delta y)^2\rangle \tag{2}$$

and related quantities as a function of the parallel displacement z, where we average over an ensemble of fluctuations $\mathbf{b}$. We typically define the ensemble as a set of fluctuations with the same correlation function (i.e., the same power spectrum). In order to concentrate on the process of FLRW evolution, here we consider the case of axisymmetric fluctuations in which $b_x(\mathbf{x})$ and $b_y(\mathbf{x})$ are statistically identical (for treatments of asymptotic diffusion for non-axisymmetric fluctuations, see Pommois et al. 1999; Ruffolo et al. 2006; Weinhorst et al. 2008) and use $V = V_x = V_y$.

Next, we can define a running diffusion coefficient as

$$D(z) \equiv \frac{1}{2}\frac{dV}{dz}. \tag{3}$$

In a limit of large z, for cases of astrophysical interest D typically approaches a constant asymptotic value, so that $V = \langle\Delta x^2\rangle \approx 2Dz$. (Note that some other studies define $D(z) = V/(2z)$, which gives the same value for asymptotic diffusion but may be different at general z.)

By integrating Equation (1), we obtain

$$\Delta x = x(z) - x(0) = \frac{1}{B_0}\int_0^z b_x[\mathbf{x}_\perp(z'), z']dz' \tag{4}$$

$$V = \langle\Delta x^2\rangle = \frac{1}{B_0^2}\int_0^z\int_0^z \langle b_x[\mathbf{x}_\perp(z'), z']b_x[\mathbf{x}_\perp(z''), z'']\rangle dz'dz'', \tag{5}$$

where $\mathbf{x}_\perp = (x, y)$ as a function of z is a (random) magnetic field line trajectory. Then it follows that

$$D = \frac{1}{B_0^2}\int_0^z \langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(\Delta z'), \Delta z']\rangle d\Delta z', \tag{6}$$

where we assume the fluctuations to be statistically homogeneous and $\mathbf{\Delta x'}_\perp(\Delta z') \equiv \mathbf{x}_\perp(z'') - \mathbf{x}_\perp(z')$ (for an illustration, see Figure 3 of Ruffolo et al. 2004). In the limit $z \to \infty$ this yields the well-known Taylor–Green–Kubo expression for the asymptotic diffusion coefficient (Taylor 1922; Green 1951; Kubo 1957). Note that $\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(\Delta z'), \Delta z']\rangle$ is a Lagrangian correlation, for which we will sometimes use the shorthand $\langle b_x b'_x\rangle$. The above relations can be expressed as a second-order ODE,

$$\frac{d^2V}{dz^2} = \frac{2}{B_0^2}\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(z), z]\rangle, \tag{7}$$

or as an equivalent system of two first-order ODEs,

$$\frac{dV}{dz} = 2D \tag{8}$$

$$\frac{dD}{dz} = \frac{1}{B_0^2}\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(z), z]\rangle. \tag{9}$$

The appropriate initial conditions are $V(0) = D(0) = 0$. Similar general ODEs have been presented by Saffman (1962) and Taylor & McNamara (1971) for situations that are mathematically analogous to the FLRW. In this form, the ODE is exact and applies to transverse fluctuations in general. The ODE is usually non-trivial because the right-hand side depends on the statistical behavior of the displacement $\mathbf{\Delta x'}_\perp$.

However, in the special case of slab fluctuations, where $\mathbf{b}$ depends only on z, the Lagrangian correlation is simply the Eulerian correlation function $R_{xx}(z)$ and the ODE is exactly integrable. The running diffusion coefficient is given by

$$D(z) = \frac{1}{B_0^2}\int_0^z R_{xx}(z')dz'. \qquad \text{(slab)} \tag{10}$$

In the limit $z \to \infty$ we recover the quasilinear expression for the asymptotic diffusion coefficient as determined by Jokipii & Parker (1968),

$$D(\infty) = \frac{\langle b_x^2\rangle}{B_0^2}\ell_c, \qquad \text{(slab)} \tag{11}$$

where ℓ_c is the correlation length of the slab fluctuations.

In the usual case where $\mathbf{b}$ depends on x and/or y, in order to develop an analytic theory of the FLRW from the ODE, a closure relation is needed to approximate the Lagrangian correlation in terms of either known quantities or $V(z)$ itself. Previous work has often used a closure in terms of Corrsin's independence hypothesis, as described below. (For cases where trapping along flux surfaces is important, an alternative to Corrsin's hypothesis is the decorrelation trajectory method (e.g., Vlad et al. 1998; Neuer & Spatschek 2006; Negrea et al. 2007). An alternative approach, based on the direct interaction approximation (Kraichnan 1959), has been shown by Vanden Eijnden & Balescu (1995) to be consistent with the present approach.) Furthermore, almost all work based on Corrsin's hypothesis has assumed a Gaussian displacement distribution (e.g., Salu & Montgomery 1977). A notable exception is Shalchi et al. (2009), who considered deviations from Gaussian field line displacements via a kappa distribution.

2.2. Closure Assuming Corrsin's Hypothesis and Gaussian Displacements

Using Corrsin's hypothesis (Corrsin 1959), the Lagrangian correlation can be approximated in terms of the Eulerian correlation function $R_{xx}(\mathbf{x}_\perp, z)$ of the magnetic fluctuation and the probability $P(\mathbf{x}_\perp|z)$ that a field line has a displacement $\mathbf{x}_\perp$ after a distance z:

$$\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(z), z]\rangle = \int R_{xx}(\mathbf{x}_\perp, z)P(\mathbf{x}_\perp|z)d\mathbf{x}_\perp. \tag{12}$$

It is useful to express the correlation function $R_{xx}(\mathbf{x}_\perp, z)$ in terms of its Fourier transform, the power spectrum $P_{xx}(\mathbf{k})$:

$$R_{xx}(\mathbf{x}_\perp, z) = \frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}d\mathbf{k}. \tag{13}$$

Then

$$\begin{aligned}\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(z),z]\rangle \\ = \frac{1}{(2\pi)^{3/2}}\iint P_{xx}(\mathbf{k})e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}P(\mathbf{x}_\perp|z)d\mathbf{x}_\perp d\mathbf{k}.\end{aligned} \tag{14}$$

Another common assumption is that the displacement distribution $P(\mathbf{x}_\perp|z)$ is Gaussian in x and y, say, with standard deviation $\sigma(z)$. Then we perform the integration over $\mathbf{x}_\perp$ to obtain

$$\langle b_x(0,0)b_x[\mathbf{\Delta x'}_\perp(z),z]\rangle = \frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}e^{-ik_z z}d\mathbf{k}, \tag{15}$$

where $k_\perp^2 = k_x^2 + k_y^2$. Then the ODE becomes

$$\frac{d^2V}{dz^2} = 2\frac{dD}{dz} = \frac{2}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}e^{-ik_z z}d\mathbf{k}. \tag{16}$$

In the next step, previous work has approximated this Lagrangian correlation in terms of either known quantities or $V(z)$ itself, which serves as closure for the ODE.

2.2.1. Integrable Closures

We refer to a closure that approximates the Lagrangian correlation in terms of known quantities as an integral closure, because then we can determine $D(z)$ by integrating Equation (9) and $V(z)$ by integrating $2D(z)$. A large number of theories in the literature are equivalent to integral closures, even if they were not expressed in terms of this ODE.

For an analogous problem, Salu & Montgomery (1977) used what we call DD, setting $\sigma(z)$ as appropriate for *asymptotic* diffusion. This approach was applied to the FLRW by Matthaeus et al. (1995). In our present notation, this assumption uses

$$\sigma^2 = 2D(\infty)z. \qquad \text{(DD)} \tag{17}$$

Then Equation (16) can be integrated over z from 0 to ∞ to obtain an expression for $D(\infty)$ that depends on $D(\infty)$ itself, that is, an implicit equation for the asymptotic diffusion coefficient. After solving that, $D(\infty)$ could be used to integrate Equation (16) to any z, to determine $D(z)$. Note that this approach takes the behavior for long z (asymptotic diffusion) to apply to all z.

Ghilea et al. (2011) introduced the assumption of RBD, in which the field lines spread ballistically in random directions:

$$\sigma^2 = \frac{\langle b_x^2\rangle}{B_0^2}z^2. \qquad \text{(RBD)} \tag{18}$$

Then Equation (16) can be integrated to determine $D(z)$, and in particular Ghilea et al. (2011) determined an explicit expression for the asymptotic diffusion coefficient. RBD theories assume that the Lagrangian correlation is significantly non-zero only at short distances z, where the field line trajectories are roughly ballistic.

That work showed that for different parameter ranges, either DD or RBD provided a closer match to direct computer simulation results, for reasons that could be understood physically but with no obvious "markers" of the range of validity. For nearly 2D turbulence, neither model worked well, which was attributed to effects of trapping along flux surfaces. While it is useful and interesting to know whether and why a given random walk problem is dominated by DD or RBD, our motivation for the present work was to find a unified approach that could naturally model the transition from short-z (RBD) to long-z (DD) behavior. We therefore turned to self-closure of the ODE, as described below.

2.2.2. ODE Self-closure

We refer to a closure that expresses the Lagrangian correlation in terms of the variance V as a self-closure, which still yields a second-order ODE. As suggested by Saffman (1962) and Taylor & McNamara (1971) and studied further by Lundgren & Pointin (1976), such a closure can self-consistently replace $\sigma^2(z)$ by $V(z)$ to yield (in our notation)

$$\frac{d^2V}{dz^2} = \frac{2}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2V(z)}e^{-ik_z z}d\mathbf{k}, \tag{19}$$

or as an equivalent system of two first-order ODEs,

$$\frac{dV}{dz} = 2D \tag{20}$$

$$\frac{dD}{dz} = \frac{1}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2V(z)}e^{-ik_z z}d\mathbf{k}. \tag{21}$$

Similar derivations were performed more recently by Wang et al. (1995) and Shalchi & Kourakis (2007). In general, the above equation does not have an analytic solution. If $P_{xx}(\mathbf{k}) \propto P_{xx}(k_\perp)\delta(k_z)$ (i.e., for a purely 2D fluctuating field $\mathbf{b}$ that does not depend on z), then one can obtain an asymptotic solution for D via a first integral (e.g., Taylor & McNamara 1971) that is $\sqrt{2}$ times bigger than D found via the DD closure of Equation (17). However, in this work we want to obtain V and D as a function of z for a more general power spectrum. We do so by solving Equation (19) numerically as the system of two first-order ODEs (Equations (20) and (21)), evaluating the Fourier integral over $\mathbf{k}$ with the QUADPACK library (Piessens et al. 1983) with initial values $V(0) = 0$ and $D(0) = 0$ to be consistent with ballistic field line trajectories ($V \propto z^2$) at low z.

In the remainder of this work, we will compare results from the ODE in Equation (19) with results from other theories and computer simulations.

3. COMPARISON WITH COMPUTER SIMULATIONS

To compare with computer simulations, we need to specify the form of the fluctuation $\mathbf{b}$ and its power spectrum. We employ a two-component 2D+slab model of transverse magnetic fluctuations in which the power spectrum is a sum of a 2D power spectrum, depending on k_x and k_y, and a slab power spectrum depending on k_z. This model is convenient for direct computer simulations because generating a realization can be done by performing only one 2D inverse fast Fourier transform (FFT) and two one-dimensional inverse FFTs, yet the resulting field $\mathbf{b}$ varies along all three dimensions. The two-component model was motivated by observations of interplanetary magnetic fluctuations, indicating quasi-slab and quasi-2D components (Matthaeus et al. 1990; Weygand et al. 2009), which can be modeled using a ratio of slab:2D fluctuation energies of approximately 20:80 (Bieber et al. 1994, 1996). This model has provided a useful description of the parallel transport of particles in the inner heliosphere (Bieber et al. 1994). By using this 2D+slab model, we can directly compare our new results from the ODE with self-closure (Equation (19)) with results from the DD (Matthaeus et al. 1995) and RBD (Ghilea et al. 2011) approaches. (Note that for RBD we use Equation (18) for the total 2D+slab fluctuation field, corresponding to the RBD/2D+slab model of Ghilea et al. 2011.)

Specifically, for the slab fluctuations we use

$$P_{xx}^{\mathrm{slab}}(k_z) = P_{yy}^{\mathrm{slab}}(k_z) = \frac{C^{\mathrm{slab}}}{[1+(k_z\lambda)^2]^{5/6}}, \tag{22}$$

where λ is the bendover length scale of the slab turbulence and C^{slab} is a normalization constant that is determined by the desired fraction of energy in the slab fluctuations. For the 2D fluctuations, the components b_x and b_y are related through a potential function $a(x, y)$ by $\mathbf{b}^{2D} = \nabla \times [a(x, y)\hat{\mathbf{z}}]$. In this work we assume that the 2D fluctuations are axisymmetric, so in $\mathbf{k}$-space we can write the magnetic power spectra as

$$P_{xx}^{2D} = k_y^2 A(k_\perp) \qquad P_{yy}^{2D} = k_x^2 A(k_\perp), \tag{23}$$

where $A(k_\perp)$ is the axisymmetric power spectrum of the potential function $a(x, y)$. We use

$$A(k_\perp) = \frac{C^{2D}}{[1+(k_\perp\lambda_\perp)^2]^{7/3}}, \tag{24}$$

where $\lambda_\perp$ is the perpendicular bendover scale and C^{2D} is a normalization constant used to set the desired fraction of 2D energy.

We perform the direct computer simulations for 2D+slab magnetic fluctuations, which are very similar to those of Ghilea et al. (2011), using a computer code based on that presented by Dalena et al. (2012). Gaussian magnetic fields are generated on grids using inverse FFT methods to obtain the desired power spectra as above. We generate slab fields on a periodic box of length $L_z = 400{,}000$ on $N_z = 2^{24}$ grid points and 2D fields on an $N_x \times N_y = 4096 \times 4096$ grid with $L_x = L_y = 100$. Then for a given set of turbulence parameters, as described below, we follow the trajectories of 1000 field lines starting at different locations in the box. One slab realization and 10 field lines are used per 2D realization (100 realizations in total). We solve the equations for the field line numerically using the fifth-order method of Cash & Karp (1990), which allows for an adaptive step size via the error estimate of the fourth-order solution. In the parallel direction z, we only determine ensemble average quantities over distances up to a fraction of a percent of the box length, to avoid periodicity effects for the slab component (see Ghilea et al. 2011). We output the mean squared displacement $\langle \Delta x^2 \rangle$ and also $\langle \Delta x^4 \rangle$ as a function of z, from which we can calculate the running diffusion coefficient $D(z) = (1/2)(d\langle \Delta x^2 \rangle/dz)$ and additionally the kurtosis

$$K \equiv \frac{\langle \Delta x^4 \rangle}{\langle \Delta x^2 \rangle^2} \tag{25}$$

as a function of z. We should expect $K = 3$ if the displacement distribution $P(\mathbf{x}_\perp|z)$ is Gaussian, and any significant deviation from 3 implies non-Gaussian statistics. We report the kurtosis as a function of z for both the x- and y-distributions, and we can use their difference (which should be zero given our assumption of axisymmetry) to estimate the uncertainty in the determination of K. The parameters that we can vary in the 2D+slab model simulations are

$$r \equiv \frac{b}{B_0}\frac{\lambda}{\lambda_\perp} \tag{26}$$

and $f_s = (b^{\mathrm{slab}}/b)^2$. The parameter r is essentially a scaled turbulence level, and f_s tells us the fraction of the turbulent energy in the slab field. Numerical results shown here are for perpendicular distances in units of $\lambda_\perp$ and parallel distances in units of λ. Note that in the case of pure slab turbulence there is no concept of $\lambda_\perp$, so r does not make sense. In that case one would instead vary b/B_0 and obtain distances in units of λ. For the results shown here, we have set $\lambda = \lambda_\perp = 1$ in all calculations.

Although r in Equation (26) looks like a Kubo number for the FLRW (such as that of Isichenko 1991a), it is conceptually different (Ghilea et al. 2011). The usual definition of the Kubo number takes into account the correlation length scales associated with the *total* field parallel and perpendicular to the mean magnetic field. Such a definition is appropriate for a single turbulence component, but with 2D+slab turbulence we have two distinct components: the 2D component, which has an infinite perpendicular correlation length (as it is independent of z), and the slab component, which has an infinite parallel correlation length (as it is independent of x and y). Therefore, the Kubo number cannot be determined for 2D+slab turbulence. Also, for understanding different regimes, r behaves differently than the Kubo number. For the one-component turbulence of Isichenko (1991a), a low Kubo number implies quasilinear behavior, whereas in 2D+slab, this corresponds to $r \gg 1$.

As a first test, we consider the FLRW evolution for pure slab fluctuations with $b/B_0 = 1$. In this case, the ODE is exactly integrable and reduces to Equation (10), so we expect nearly perfect agreement between the ODE and simulation results (as previously obtained by Shalchi & Qin 2010). Such agreement is demonstrated in Figure 1 and gives us confidence in our methodology. The Lagrangian correlation $\langle b_x b_x' \rangle$ shown in Figure 1(b) is simply the Eulerian correlation associated with our input power spectrum, and this is integrated over z to yield the running diffusion coefficient D_x in Figure 1(a). The horizontal dashed line in Figure 1(a) is the prediction of quasilinear theory for the asymptotic field line diffusion coefficient (Jokipii & Parker 1968) of $D_x(\infty) = 0.384$, which matches very closely with the simulation result of 0.385. Note that the kurtosis K determined by the simulation shown in Figure 1(c) is equal to 3, within the uncertainty, which is consistent with Gaussian distributions for the field line displacement at all z. Note that in this example and those that follow, we only show the part of the result over which the computer simulation quantities undergo significant evolution; at later z, D is *diffusive* in all cases and does not change significantly from the last shown value (K does not change much either). We provide asymptotic values for D in the text, which are obtained at z at least 10 times beyond the last plotted values. We use a very large simulation box because we can sometimes see the start of periodicity effects due to the slab turbulence at z not much larger than the range of z where asymptotic values are determined.

Next, we show an example for 2D+slab fluctuations with $r = 1$ and a slab fraction $f_s = 0.8$ (see Figure 2). This is a case where Ghilea et al. (2011) found that the asymptotic diffusion coefficient from DD matched that from simulations better than RBD, but neither matched very well (or very poorly). Figure 2(a) shows that the ODE with self-closure (Equation (19)) can very closely match simulation results for the evolution of D_x as a function of the parallel displacement z. The results for DD and RBD do not match the evolution of $D(z)$ as well. Interestingly, the Lagrangian correlation as plotted in Figure 2(b) appears visually similar for the various approximations (for the ODE,

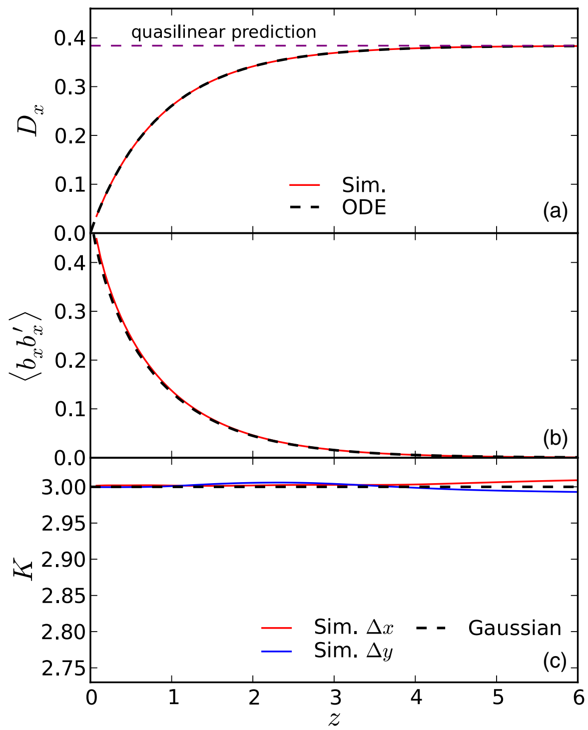

Figure 1. (a) Evolution of the running magnetic field line diffusion coefficient D_x as a function of the displacement z along the large-scale field for pure slab turbulence with $b/B_0 = 1$ according to the ODE discussed in the text (dashed black curve) and direct computer simulations (red curve). The horizontal dashed line indicates the asymptotic diffusion coefficient derived from quasilinear theory (Jokipii & Parker 1968), which is very accurate for pure slab turbulence. (b) The Lagrangian correlation function $\langle b_x b_x' \rangle \equiv \langle b_x[\mathbf{x}_\perp(0), 0] b_x[\mathbf{x}_\perp(z), z] \rangle$. (c) The kurtosis of the distributions of the displacements Δx (red curve) and Δy (blue curve), which remains close to the value of 3 as expected for a Gaussian displacement distribution. This figure indicates that for pure slab turbulence, the field line random walk evolution is very well explained by the ODE, as expected because the decorrelation of the fluctuating field is independent of the distributions of displacements Δx and Δy.

(A color version of this figure is available in the online journal.)

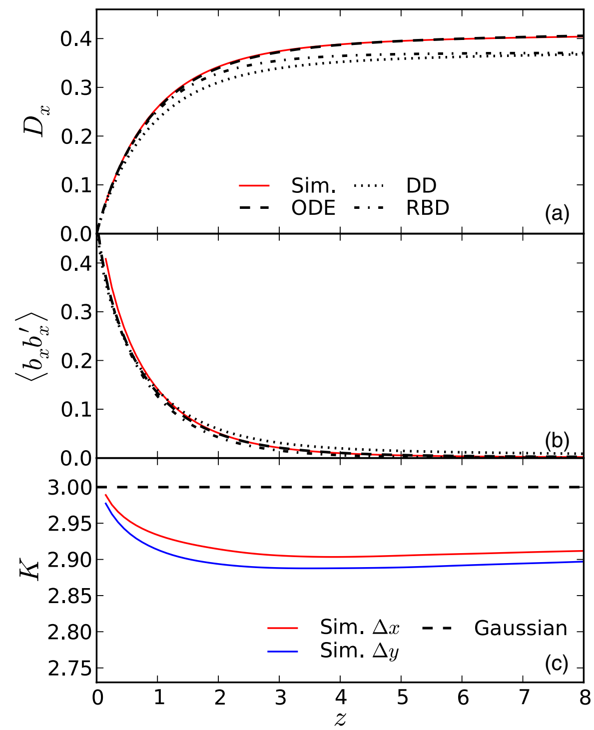

Figure 2. Like Figure 1, but for 2D+slab turbulence with $r = 1$ and a slab fraction $f_s = 0.8$. Results are shown for direct computer simulations (red curve), the ODE (dashed black curve), and the assumptions of diffusive decorrelation (DD; dotted curve) and random ballistic decorrelation (RBD; dash-dotted curve). All model assumptions yield results in reasonable agreement with computer simulations, and the ODE model provides the best agreement. The kurtosis remains above 2.8, which is found to be associated with good agreement between the ODE model and computer simulations.

(A color version of this figure is available in the online journal.)

DD, or RBD) and the simulation, yet the differences for DD and RBD become apparent when integrating to obtain $D(z)$.

Let us examine whether the ODE with self-closure has served to model the transition from RBD behavior at low z to DD behavior at high z, which was the motivation for this study. In Figure 2(a) it is seen that for $z < 1$, the RBD theory provides a very close match to the ODE and to simulation results, whereas the DD model deviates noticeably. However, at higher z the evolution of $D(z)$ from the RBD theory flattens out too quickly and increasingly deviates from the ODE and simulation results. On the other hand, while $D(z)$ from the DD model deviates initially, at later z it evolves with a very similar slope to that for the ODE and simulations. Finally, for asymptotic diffusion the results for the ODE, simulation, DD, and RBD are 0.427, 0.419, 0.390, and 0.371, respectively. We see that the ODE indeed provides a better match with simulations, not only of the asymptotic diffusion coefficient but also of the FLRW evolution, thus increasing confidence that it models the physics better by naturally transitioning from RBD to DD behavior as a function of z.

From Figure 2(c), we see that the kurtosis K is significantly different from 3 throughout this z-range, indicating that the displacement distribution has a significant deviation from Gaussianity. However, from this and other examples we have found that when K remains above about 2.8, the ODE provides an accurate estimate of the diffusion coefficient $D(z)$.

Finally, we consider an example with $r = 0.3$ and $f_s = 0.4$, for which Ghilea et al. (2011) found that the DD theory provided a very good match to the simulation value for the asymptotic diffusion coefficient. Figure 3(a) shows that none of the theories provide an accurate description of the FLRW evolution for $z > 10$, though the ODE remains accurate for the greatest distance z, and the DD model approaches the simulation result asymptotically. Similarly, from Figure 3(b) we may conclude that none of the theories provide a very accurate description of the Lagrangian correlation.

Interestingly, the kurtosis K falls below 2.8, and from a survey over parameter space (A. P. Snodin et al. 2012, in preparation) we find that such a low minimum K value is associated with a substantial deviation of the ODE from the simulation results. To put that another way, sufficiently non-Gaussian displacement distributions are associated with inaccuracy of the ODE. An examination of the displacement distribution at the minimum kurtosis found in Figure 3(c) confirms that it is non-Gaussian (see Figure 4). It can easily be shown that a sum of two different Gaussian distributions will always have $K > 3$, which is not consistent with our simulation results.

The asymptotic diffusion coefficients are 0.133, 0.103, 0.102, and 0.757 for the ODE, DD, simulation, and RBD, respectively. From the asymptotic diffusion coefficient alone, one might conclude that the DD theory works well. However, from examination of the evolution in Figure 3, we must conclude that the DD theory fails to correctly describe the evolution of

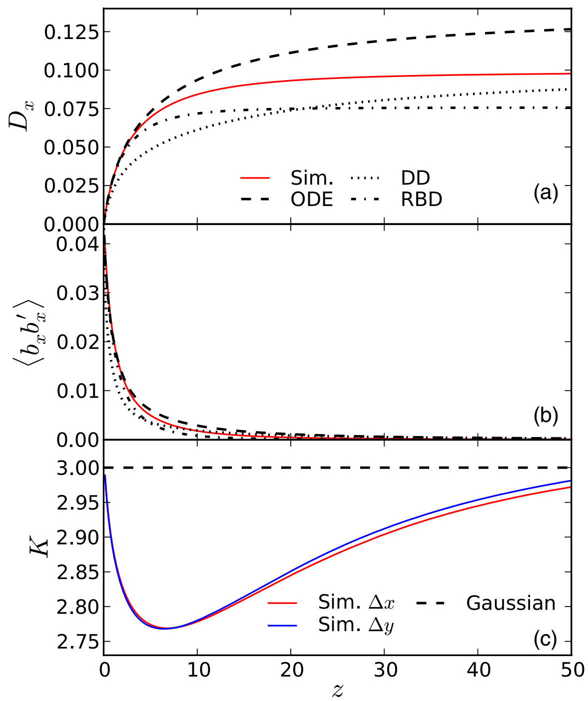

Figure 3. Like Figure 1, but for 2D+slab turbulence with $r = 0.3$ and a slab fraction $f_s = 0.4$. Results are shown for direct computer simulations (red curve), the ODE (dashed black curve), and the assumptions of diffusive decorrelation (DD; dotted curve) and random ballistic decorrelation (RBD; dash-dotted curve). In this case, all models deviate substantially from the simulation results for the field line random walk evolution, presumably because of trapping effects. The minimum kurtosis is below 2.8, implying a displacement distribution that is substantially non-Gaussian. Such a low minimum kurtosis is found to be associated with substantial deviation between the ODE model and computer simulations.

(A color version of this figure is available in the online journal.)

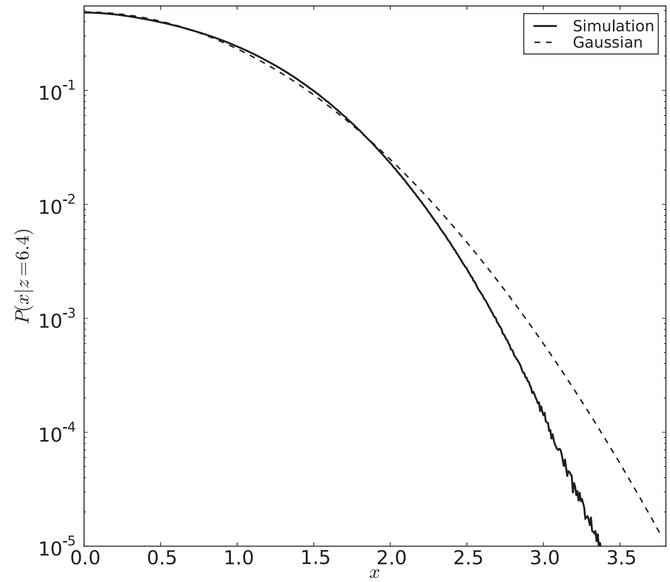

Figure 4. PDF of displacements x from computer simulations found at the minimum kurtosis (i.e., at about $z = 6.4$) of Figure 3(c) ($r = 0.3$, $f_s = 0.4$). The PDF is shown only for $x \geqslant 0$. Also shown is a Gaussian PDF with the same variance as found in the simulation.

$D(z)$, despite its near match with the simulation for asymptotic diffusion. In particular, the ODE starts to become inaccurate at $z \sim 3$, and from then on its estimate of $D(z)$ is too large, continuing to increase with z even when the simulation value has become nearly constant. The assumption of DD, which is designed for large z values, becomes inaccurate much earlier (at $z \sim 1$) and underestimates $D(z)$. However, like the ODE, its estimate of $D(z)$ continues to rise with z, which compensates for its initial underestimate, bringing it into closer agreement.

4. DISCUSSION

Our motivation for this work was to seek a unified theory of the FLRW that can provide improved accuracy by treating a transition from low-z (RBD) to high-z (DD) behavior and to verify this transition by examining the evolution of the FLRW with distance z. We have considered the ODE introduced by Saffman (1962), Taylor & McNamara (1971), and others and explain how a variety of FLRW theories in the literature are related to this ODE. It is exactly integrable for the case of pure slab fluctuations. For other cases, various studies have used closures to relate the Lagrangian correlation to known quantities or the variance $V(z)$, often employing Corrsin's hypothesis (Corrsin 1959) and assuming a Gaussian displacement distribution.

We have employed a version of the ODE with a closure that self-consistently calculates the variance $V(z)$ of the magnetic field line displacement distribution as a function of distance z along the large-scale field. The results from this ODE were compared with RBD and DD models, as well as direct computer simulations of the FLRW. We examined the FLRW evolution, and for the cases considered, all theories and simulations are consistent with the expected free-streaming (random ballistic) trajectories at low z, for which $V(z) \approx (\langle b_x^2 \rangle / B_0^2) z^2$ and $D(z) \approx (\langle b_x^2 \rangle / B_0^2) z$. The theories and simulations then approach asymptotic diffusion at large z, but with values of the asymptotic diffusion coefficient that depend on details of the intermediate evolution.

We find that in some cases the ODE provides a better match than RBD or DD to direct simulation results, not only for asymptotic diffusion but also for the FLRW evolution, thus increasing confidence that it correctly models the physics underlying the field line diffusion. From the details of the evolution, we can also verify that the ODE behaves like the RBD model at low z and like the DD model at high z (see Figure 2). In such cases, we have successfully identified an accurate model that treats the transition from RBD to DD behavior.

That said, it should be noted that for practical applications the ODE is more complicated to use. To include the field line diffusion coefficient in another theory or computer code, the explicit formula for RBD (Ghilea et al. 2011) or even the implicit formula for DD (Matthaeus et al. 1995) is more convenient. Also, when the ODE is accurate, the RBD and DD theories also match simulations to within a factor of two (A. P. Snodin et al. 2012, in preparation). For some applications that level of accuracy might be sufficient, considering that some of the inputs to a calculation such as the fluctuation energy and the bendover scales λ and $\lambda_\perp$ may also be uncertain or variable.

For the case shown in Figure 3, why is the ODE inaccurate? This could be due to the assumption of Corrsin's hypothesis or the assumption of a Gaussian displacement distribution, and further evidence is needed to distinguish between those possibilities. We do find that substantial non-Gaussianity (i.e., low kurtosis) is associated with inaccuracy of the ODE, but that does not necessarily imply that it causes the inaccuracy—a point that is quite easy to recognize from the work of Weinstock (1976).

Ghilea et al. (2011) identified trapping behavior as a reason for inaccuracy of the RBD and DD theories (of Bohm diffusion) in some parameter ranges. Trapping behavior and/or percolation (e.g., Gruzinov et al. 1990; Isichenko 1991b; Ottaviani 1992) relate to temporary topological trapping on closed flux surfaces when the fluctuations are nearly 2D, or more precisely, when the asymptotic field line diffusion is dominated by the 2D component of fluctuations. Such trapping can also explain dropouts of energetic particles from impulsive solar flares as observed near Earth (Ruffolo et al. 2003; Chuychai et al. 2007; Seripienlert et al. 2010). Although for the case shown in Figure 3, the asymptotic diffusion coefficient for DD agrees with that from simulation results, so that Ghilea et al. (2011) did not include this case in the parameter range with trapping effects, we show that the DD theory does not accurately model the FLRW evolution and thus conclude that the agreement is fortuitous. Given that both RBD and DD provide inaccurate descriptions of the evolution, and that the ODE provides a transition from RBD to DD behavior, we conclude that all three theories are probably failing due to trapping effects. Some previous studies already concluded that trapping behavior implies that Corrsin's hypothesis does not work properly (Vlad et al. 1998; Neuer & Spatschek 2006; Negrea et al. 2007), which is a possible explanation for why all these theories are inaccurate for this case.

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