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Theory of magnetic field line random walk in noisy reduced magnetohydrodynamic turbulence

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When a magnetic field consists of a mean part and fluctuations, the stochastic wandering of its field lines is often treated as a diffusive process. Under suitable conditions, a stable value is found for the mean square transverse displacement per unit parallel displacement relative to the mean field. Here, we compute the associated field line diffusion coefficient for a highly anisotropic "noisy" reduced magnetohydrodynamic model of the magnetic field, which is useful in describing low frequency turbulence in the presence of a strong applied DC mean magnetic field, as may be found, for example, in the solar corona, or in certain laboratory devices. Our approach is nonperturbative, based on Corrsin's independence hypothesis, and makes use of recent advances in understanding factors that control decorrelation over a range of parameters described by the Kubo number. Both Bohm and quasilinear regimes are identified.
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I. INTRODUCTION

The magnetic field line random walk (FLRW) is an important feature of magnetic fields in space and astrophysical settings, describing the statistical character of the field itself, and also providing an important step in understanding transport of energy, such as electron heat flux and confinement of energetic charged particles.[1–3] For these reasons, the FLRW is a nearly ubiquitous feature that influences magnetic connectivity, a concept that is frequently encountered in mapping field lines from the Sun to the geospace environment, or from an emitter of particles to a spacecraft or even a ground level detector.[4–6] On general grounds, one expects for a homogeneous system that the mean square displacement of field lines $\langle(\Delta x)^2\rangle$ becomes diffusive with a constant, asymptotic diffusion coefficient $D_\infty = \langle(\Delta x)^2\rangle/(2z)$ in the limit of large displacement z along the mean (or average) magnetic field. There are interesting cases in which this expectation is not achieved, notably when too much symmetry is present, for a periodic system, or in cases where trapping can occur, and in such cases the mean square displacement may scale as z^β with $\beta \neq 1$.[7] However, for a very broad class of homogeneous systems with finite correlation scales ℓ_c, we expect that when $z \gg \ell_c$, the increments of the displacement become uncorrelated and $\langle(\Delta x)^2\rangle$ becomes diffusive. It is such cases that concern us here. We will work in the context of the reduced magnetohydrodynamics (RMHD) model in which the magnetic field is composed of a mean field and transverse fluctuations.[8] Specifically, we will derive a diffusion coefficient for FLRW in the context of homogeneous "noisy" RMHD (to be explained shortly), employing nonperturbative methods based on the Taylor-Green-Kubo (TGK) formulation of diffusion and the use of Corrsin's independence hypothesis. Along the way, we will employ recent developments in understanding how decorrelation of the TGK integrand occurs at large displacements, without the usual assumption of quasilinear ordering. In this way, the theory will be intrinsically nonlinear, and for various ranges of Kubo number (see below) we will find both quasilinear and Bohm-like regimes of diffusion. In Sec. VI, we will outline several potential applications of this analytical framework for field line transport in RMHD.

II. BACKGROUND AND RECENT DEVELOPMENTS IN FLRW DIFFUSION THEORY

For definiteness, we view the magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ as composed of a mean field $\mathbf{B}_0$ and a fluctuating component $\mathbf{b}$. An averaging operator defines these according to $\mathbf{B}_0 \equiv \langle\mathbf{B}\rangle$, and we assume that the fluctuations are transverse, meaning that $\mathbf{b}\cdot\mathbf{B}_0 = 0$. We do not assume that fluctuations are small as we will not be carrying out a perturbation theory. However, in order to assert the validity of the transversality condition, we generally expect that the mean field strength B_0 cannot be too small compared with the root mean square fluctuation b. (A still stronger condition is needed for RMHD—see below.) In fact, transversality, equivalent to suppressing the parallel variance of the fluctuations, is a basic feature of highly anisotropic models of MHD turbulence,[8–10] and occurs most readily in models that are weakly compressive.[11]

Apart from the general expectation of diffusive behavior at large displacements in homogeneous systems with finite correlation scale as alluded to above, there are three broad types of FLRW behavior that have been described: (1) In the quasilinear limit,[1,12] the diffusion coefficient scales as $D_\infty \sim (b/B_0)^2$. This is associated with the limit of low R, where one defines the Kubo number as

$$R = \frac{b}{B_0}\frac{\lambda_\parallel}{\lambda_\perp}. \tag{1}$$

Other possibilities are (2) so-called Bohm diffusion with $D_\infty \sim b/B_0$, which has been extensively studied in the context of laboratory devices[13,14] and is generally associated with high Kubo number $R \gg 1$; and (3) percolation related to trapping along 2D flux surfaces[3,15] which also can occur at high Kubo number R.

Many, but not all theories of FLRW in transverse turbulence are based on the Taylor-Green-Kubo formula,[16–18] which can be written for an (x, y, z) Cartesian system aligned with $\mathbf{B}_0 = B_0\hat{z}$ as

$$\frac{\langle(\Delta x)^2\rangle}{2\Delta z} = \int_0^\infty \langle b_x[\mathbf{x}_\perp(0), 0] b_x[\mathbf{x}_\perp(z), z]\rangle dz, \qquad (2)$$

where the integrand is the Lagrangian correlation $R_L(z) = \langle b_x[\mathbf{x}_\perp(0), 0] b_x[\mathbf{x}_\perp(z), z]\rangle$, having randomly distributed arguments $\mathbf{x}_\perp$. Care should be taken to not confuse R_L with the Eulerian correlation $R(x, y, z) = \langle b_x(0, 0, 0) b_x(x, y, z)\rangle$, which has fixed (non-statistical) arguments. In both cases, the origin is arbitrary due to the assumption of spatial homogeneity.

In the classic quasilinear theory of the FLRW, due to *Jokipii*[12] and *Jokipii and Parker*,[1] one approximates the diffusion coefficient by assuming that one can ignore the dependence on the random argument in the integral, in effect setting $\mathbf{x}_\perp(z) \to 0$. More recent FLRW work has focused on several issues and extensions, notably including variation of the Kubo number to investigate non-quasilinear behavior, and adoption of models for the turbulence that go beyond the one-dimensional "slab" field that occupies a central role in the quasilinear case. Extensions to FLRW have often proceeded through considerations of the dependence of the TGK integral on the trajectory. When a random trajectory is included on the right hand side of Eq. (2), the theory can become nonlinear.

A substantial amount of work on the FLRW as well as particle scattering theory has made use of a "two-component" 2D+slab model of transverse turbulent fluctuations, which was motivated by observations[19] and supported at a basic physics level by simulations.[20,21] In this model, the slab component $\mathbf{b}^{\rm slab}(z)$ has wavevectors only along the k_z-axis, with $k_x = k_y = 0$, and the 2D component $\mathbf{b}^{\rm 2D}(x, y)$ has wavevectors only in the k_x-k_y plane, with $k_z = 0$. The magnetostatic 2D+slab model is sufficiently simple to permit tractable analytic theories. The model includes power in all dimensions of **k**-space but only requires one- and two-dimensional Fourier transforms to create synthetic realizations of turbulence, so it provides a useful testbed for direct computer simulations of turbulent field line diffusion to test the analytic theories. The first analytic theory for the 2D+slab FLRW[22] proposed a combination of quasilinear and Bohm diffusion terms. In fact, computer simulations of the FLRW in 2D+slab turbulence have found evidence for all 3 types of behavior in different ranges of the parameters for the turbulence amplitude and slab fraction.[23]

Corrsin's independence hypothesis[24] has often been employed in treatments of the FLRW in non-perturbative models.[14,22,25] This is a simple but powerful approximation, to be discussed further below, that replaces the Lagrangian correlation by a statistical sampling of the Eulerian correlation. Once Corrsin's hypothesis is adopted, the relevant question becomes how to approximate the statistics of the trajectory in the approximate TGK integrand, especially the variance $\sigma^2(z)$ of the conditional probability distribution $P(\mathbf{x}_\perp, z)$. One way to obtain Bohm-like diffusion terms is to use what we call "diffusive decorrelation" (DD), setting $\sigma^2(z) = 2D_\infty z$, as appropriate for asymptotic diffusion. A more recent theory based on random ballistic decorrelation (RBD) considered that two-component field lines spread ballistically in random directions with $\sigma^2(z) = (\langle b_x^2\rangle/B_0^2)z^2$, which is relevant for the z-range before the fluctuating field decorrelates and the field line trajectories become diffusive.[23] Each theory, using DD or RBD, was found to provide a better description of simulation results in different parameter regimes, due to the interplay of the 2D and slab components of turbulence.

While the two-component model (2D+slab model) of magnetic fluctuations has certain advantages, there are some disadvantages. With its simplified structure, the 2D+slab model has no oblique modes at general wavevectors **k**. In addition, the Kubo number is undefined, making it difficult to relate results for 2D+slab turbulence, with extended regions of successful agreement between Corrsin-hypothesis theories (which combine quasilinear and Bohm diffusion) and simulation results, to results for other turbulence models as expressed in terms of the Kubo number.

There have also been numerical studies of the FLRW or analogous problems using single-component models of turbulence,[7,26–28] which described a transition from quasilinear to percolative behavior and also transitions between diffusive and super- or sub-diffusive behavior. More recently, the local power-law index γ (such that $D_\infty \propto R^\gamma$) of the FLRW was reported to vary from the quasilinear value of $\gamma = 2$ at very low Kubo number to roughly the percolation value of $\gamma = 0.7$ at extremely high Kubo number.[29] Given that observed plasmas typically have some coupling between quasi-slab and quasi-2D modes that exchanges energy between them, extreme Kubo numbers are unlikely to be realized in nature, so it is still quite interesting to consider the range $0.1 < R < 10$, in which their results may also be consistent with a combination of quasilinear and Bohm diffusive behavior.

Here, we present non-perturbative analytic calculations based on Corrsin's hypothesis for the case of magnetic fluctuations of the type found in RHMD turbulence. There are several reasons that motivate this choice for the magnetic field model. RMHD is a widely used model for describing anisotropic magnetic turbulence in the presence of a strong large-scale magnetic field. For RMHD fluctuations, the Kubo number is well-defined, and the magnetic fluctuations are transverse. These properties provide substantial simplification for both theory and computer simulations. RMHD is a dynamical model, so that understanding of the FLRW for magnetostatic RMHD fluctuations can serve as a baseline for studying dynamical effects.

In the present work, we use what we call a "noisy RHMD" model, in which the range K of the wavevector component k_z along the large-scale field direction $\hat{\mathbf{z}}$ is an adjustable parameter. Although RMHD is typically used for $R \sim 1$, a change in B_0 leads to a change in R. However, anisotropic turbulence with strongly transverse fluctuations is unlikely to persist if $B_0 \ll b$, making it physically unlikely that RMHD would be applicable for $R \gg 1$. In any case, our model formally allows an arbitrary specification of the Kubo number R, ranging from $R \gg 1$ with a nearly 2D FLRW to a "noisy" (quasilinear) limit with $R \ll 1$ that resembles a 1D slab fluctuation model. This flexible specification of R allows us to address the theoretical issue of the dependence of D_∞ on R, even if RMHD fields are more likely to be found in physical situations with $R \lesssim 1$.

III. NOISY RMHD MODEL OF MAGNETIC FLUCTUATIONS

The noisy RMHD model describes magnetic fluctuations with a turbulent power spectrum similar to that obtained from a dynamical RMHD simulation. We express the total magnetic field as

$$\mathbf{B} = B_0\hat{\mathbf{z}} + \mathbf{b}(x, y, z), \tag{3}$$

where $B_0\hat{\mathbf{z}}$ is a constant mean (large-scale) field and $\mathbf{b} \perp \hat{\mathbf{z}}$. The statistically homogeneous fluctuating field $\mathbf{b}$ is given by

$$\mathbf{b}(x, y, z) = \nabla_\perp \times [a(x, y, z)\hat{\mathbf{z}}], \tag{4}$$

where the subscript "$\perp$" indicates a projection perpendicular to the mean field in which only x- and y-components are included. We refer to the scalar a as the potential function. In terms of wavevectors $\mathbf{k}$, we can write

$$\mathbf{b}(\mathbf{k}) = -i\mathbf{k}_\perp \times [a(\mathbf{k})\hat{\mathbf{z}}], \tag{5}$$

and we specify the potential function in $\mathbf{k}$-space by

$$a(\mathbf{k}) \propto \begin{cases} a^{2D}(k_x, k_y)e^{i\varphi(\mathbf{k})} & \text{for } |k_z| \le K \\ 0 & \text{for } |k_z| > K, \end{cases} \tag{6}$$

where $\varphi(\mathbf{k})$ is a random phase and the proportionality constant is set by requiring the fluctuation energy to be independent of K. In terms of the power spectra, this model gives

$$P_{xx}(\mathbf{k}) = \begin{cases} \sqrt{2\pi}\, k_y^2 A(k_x, k_y)/(2K) & \text{for } |k_z| \le K \\ 0 & \text{for } |k_z| > K \end{cases}$$
$$P_{yy}(\mathbf{k}) = \begin{cases} \sqrt{2\pi}\, k_x^2 A(k_x, k_y)/(2K) & \text{for } |k_z| \le K \\ 0 & \text{for } |k_z| > K, \end{cases} \tag{7}$$

where P_{ii} is the 3D power spectrum of b_i and A is the 2D power spectrum of a^{2D}. This "boxcar" dependence on k_z has been used to characterize the results of RMHD simulations.[30] Note that this specification includes the energy-containing range, so there is no justification for adopting more specific forms, such as one appropriate for steady inertial range spectra.

It is straightforward to derive that the correlation length along z is given by $\ell_c = \pi/(2K)$. For convenience, let us assume axisymmetry around the mean field direction, i.e., that all quantities are statistically identical in the x and y directions, and $A = A(k_\perp)$ where $k_\perp \equiv \sqrt{k_x^2 + k_y^2}$. Then, the total magnetic power spectrum $P(k_\perp, k_z) \equiv P_{xx}(\mathbf{k}) + P_{yy}(\mathbf{k})$ is also axisymmetric. Now, let us specify the Kubo number as

$$R = \frac{b}{B_0}\frac{\ell_c}{\ell_\perp} = \frac{b}{B_0}\frac{\pi}{2K\ell_\perp}, \tag{8}$$

where $b \equiv \sqrt{\langle b^2 \rangle}$ represents the rms fluctuation and $\ell_\perp$ is the correlation length for the total correlation $\langle \mathbf{b}(0) \cdot \mathbf{b}(\mathbf{x}_\perp) \rangle$. It is straightforward to show that $\ell_\perp$ for the three-dimensional noisy RMHD field is the same as the total correlation length[31] λ_{c2} of the two-dimensional field corresponding to a^{2D}, so that

$$\ell_\perp = \lambda_{c2} = \frac{\int k_\perp A(k_\perp) d\mathbf{k}_\perp}{\int k_\perp^2 A(k_\perp) d\mathbf{k}_\perp} = \frac{\int P(k_\perp, k_z)/k_\perp d\mathbf{k}}{\int P(k_\perp, k_z) d\mathbf{k}}, \tag{9}$$

where $\mathbf{k}_\perp = (k_x, k_y)$. Thus, $\ell_\perp$ is the $k_\perp^{-1}$ moment of the total magnetic power spectrum.

IV. ANALYTIC THEORIES OF THE ASYMPTOTIC FIELD LINE DIFFUSION COEFFICIENT

In this section, we present two analytic derivations for the asymptotic diffusion coefficient D_∞ of the magnetic field line random walk in noisy RMHD turbulence, based on the DD and RBD approximations.

A. Diffusive decorrelation

Our derivation initially follows previous derivations for 2D+slab turbulence.[22,23,32] For transverse fluctuations, the trajectory of a magnetic field line can be described by $x(z)$ and $y(z)$, which are determined by

$$\frac{dx}{dz} = \frac{b_x(x, y, z)}{B_0}, \quad \frac{dy}{dz} = \frac{b_y(x, y, z)}{B_0}. \tag{10}$$

Considering the field line displacements Δx and Δy over a parallel displacement z, the problem of the FLRW is to then find the mean squared displacement (variance)

$$V(z) \equiv \langle (\Delta x)^2 \rangle = \langle (\Delta y)^2 \rangle, \tag{11}$$

where we have used the assumption of axisymmetry.

The change in, say, the x-coordinate over a distance z is

$$\Delta x \equiv x(z) - x(0) = \frac{1}{B_0}\int_0^z b_x[\mathbf{x}_\perp(z'), z']dz'. \tag{12}$$

The ensemble average of $(\Delta x)^2$ can be expressed by

$$\langle \Delta x^2 \rangle = \frac{1}{B_0^2}\int_0^z \int_0^z \langle b_x[\mathbf{x}_\perp(z'), z']b_x[\mathbf{x}_\perp(z''), z''] \rangle dz' dz''. \tag{13}$$

With the assumption of statistical homogeneity, one can write

$$\langle \Delta x^2 \rangle = \frac{1}{B_0^2}\int_0^z \int_{-z'}^{z-z'} \langle b_x(0, 0) b_x[\Delta\mathbf{x}'_\perp(\Delta z'), \Delta z'] \rangle d\Delta z' dz', \tag{14}$$

where $\Delta\mathbf{x}'_\perp \equiv \mathbf{x}_\perp(z'') - \mathbf{x}_\perp(z')$ and $\Delta z' \equiv z'' - z'$ for locations along a field line trajectory.

In this section, we consider asymptotic field line diffusion for large z, which occurs when the correlation vanishes after a very long distance, so we can extend the limits of the $\Delta z'$ integration to $\pm\infty$. In this case, the z' integration is trivial and we obtain

$$\langle \Delta x^2 \rangle = 2D_\infty z, \tag{15}$$

where

$$D_\infty = \frac{1}{B_0^2}\int_0^\infty \langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle dz, \tag{16}$$

which is the TGK formula for the asymptotic diffusion coefficient. This integral is dominated by low values of z where the Lagrangian correlation remains strong, before the magnetic fluctuation decorrelates.

Our first key assumption is Corrsin's independence hypothesis.[24] This assumption relates the Lagrangian correlation function in Eq. (16) to the Eulerian correlation function, $R_{xx} \equiv \langle b_x(0,0)b_x(\mathbf{x}_\perp, z)\rangle$, averaged using the conditional probability $P(\mathbf{x}_\perp|z)$ of finding a displacement $\mathbf{x}_\perp$ after a distance z

$$\langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle = \int R_{xx}(\mathbf{x}_\perp, z)P(\mathbf{x}_\perp|z)d\mathbf{x}_\perp. \tag{17}$$

Now is useful to express the correlation function $R_{xx}(\mathbf{x}_\perp, z)$ in terms of its Fourier transform, the power spectrum $P_{xx}(\mathbf{k})$

$$R_{xx}(\mathbf{x}_\perp, z) = \frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}d\mathbf{k}, \tag{18}$$

from which we obtain

$$\langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle = \frac{1}{(2\pi)^{3/2}}\int\int P_{xx}(\mathbf{k})e^{-ik_x x}e^{-ik_y y} \times e^{-ik_z z}P(\mathbf{x}_\perp|z)d\mathbf{x}_\perp d\mathbf{k}. \tag{19}$$

Our second key assumption is that the displacement distribution $P(\mathbf{x}_\perp|z)$ is Gaussian in x and y, with standard deviation $\sigma(z)$. Then, we perform the integration over $\mathbf{x}_\perp$ to obtain

$$\langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle = \frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}e^{-ik_z z}d\mathbf{k}, \tag{20}$$

where $k_\perp^2 = k_x^2 + k_y^2$. The combination of the assumptions of Corrsin's hypothesis and a Gaussian displacement distribution was employed for purely 2D turbulence.[14] Substituting this into the TGK formula, Eq. (16), we obtain

$$D_\infty = \frac{1}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})\left[\int_0^\infty e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}e^{-ik_z z}dz\right]d\mathbf{k}. \tag{21}$$

The third key assumption specifies σ^2 as a function of z. In this subsection, we use

$$\sigma^2 = 2D_\infty z, \tag{22}$$

where D_∞ is identified with the asymptotic diffusion coefficient in Eq. (16), providing closure. We refer to this framework as DD because the decorrelation of $\langle b_x(0,0)b_x[\Delta\mathbf{x}'_\perp(z),z]\rangle$ is influenced by diffusive spreading of the displacement distribution $P(\mathbf{x}_\perp|z)$ according to Eqs. (17) and (22). This DD approach was has been employed for purely 2D turbulence[25] or the 2D component of 2D+slab turbulence.[22] While Eq. (22) is evidently applicable to asymptotic diffusion, it can be inaccurate for the initial range of z.

Then, we perform the z-integration in Eq. (21) to obtain

$$D_\infty = \frac{1}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int \frac{P_{xx}(\mathbf{k})}{D_\infty k_\perp^2 + ik_z}d\mathbf{k}. \tag{23}$$

Note that the imaginary part of the integrand is an odd function of k_z, so it integrates to zero and the result is a real number. We then use the power spectrum for noisy RMHD turbulence from Eq. (7) to obtain

$$D_\infty = \frac{1}{B_0^2}\frac{1}{2\pi}\int \frac{k_y^2 A(k_\perp)}{2K}\left[\int_{-K}^{K}\frac{dk_z}{D_\infty k_\perp^2 + ik_z}\right]d\mathbf{k}_\perp. \tag{24}$$

This leads to

$$D_\infty = \frac{1}{2B_0^2}\frac{1}{2\pi}\int \frac{k_\perp^2 A(k_\perp)}{K}\tan^{-1}\left(\frac{K}{k_\perp^2 D_\infty}\right)d\mathbf{k}_\perp, \tag{25}$$

where we have made use of axisymmetry to replace k_y^2 inside the integral with $(k_x^2 + k_y^2)/2 = k_\perp^2/2$.

Note that the mean squared magnetic fluctuation $\langle b^2\rangle = R_{xx}(0) + R_{yy}(0)$, which we write, as above, as b^2, is the inverse transform of $P(k_\perp, k_z)$ at the origin. Thus

$$b^2 = \frac{1}{(2\pi)^{3/2}}\int P(k_\perp, k_z)d\mathbf{k} = \frac{1}{2\pi}\int k_\perp^2 A(k_\perp)d\mathbf{k}_\perp \tag{26}$$

and we can write the expression for D_∞ as

$$D_\infty = \frac{b^2}{2B_0^2}\frac{\int k_\perp^2 A(k_\perp)\tan^{-1}(K/k_\perp^2 D_\infty)/K d\mathbf{k}_\perp}{\int k_\perp^2 A(k_\perp)d\mathbf{k}_\perp}. \tag{27}$$

Note that this is an implicit equation in which D_∞ also appears on the right hand side.

It is useful to re-write this result in terms of scaled, dimensionless variables, indicated by primed quantities. Consider scaling quantities depending on z by $z' = Kz$, $k'_z = k_z/K$, etc., and quantities depending on x or y by $x' = x/\ell_\perp$, $k'_\perp = k_\perp\ell_\perp$, etc. This leads to a scaled diffusion coefficient $D' = D/(K\ell_\perp^2)$. Recalling from Eq. (8) that $R = (b/B_0)\pi/(2K\ell_\perp)$, we obtain

$$D'_\infty = \frac{2}{\pi^2} R^2 \frac{\int k'^2_\perp A(k'_\perp) \tan^{-1}(1/k'^2_\perp D'_\infty) d\mathbf{k}'_\perp}{\int k'^2_\perp A(k'_\perp) d\mathbf{k}'_\perp}. \tag{28}$$

These expressions for D_∞ and D'_∞ simplify in the limit of very low or very high Kubo number. Consider low R values such that $D_\infty \ll K/k_\perp^2$ over the range of $k_\perp$ where $k_\perp^2 A(k_\perp)$ is relatively large, i.e., the energy-containing range of the turbulence. In that range, the inverse tangent is approximately $\pi/2$. Recalling that $\ell_c = \pi/(2K)$, we obtain

$$D_\infty = \frac{1}{2}\frac{b^2}{B_0^2}\ell_c \quad (\text{low}\, R), \tag{29}$$

which is precisely the quasilinear result.[1] In terms of the dimensionless diffusion coefficient, this becomes

$$D'_\infty = \frac{R^2}{\pi} \quad (\text{low}\, R) \tag{30}$$

and we can see that D'_∞ varies as the Kubo number squared.

Now, consider high R values such that $D_\infty \gg K/k_\perp^2$ over the energy-containing range. In this range, $\tan^{-1}x \approx x$ and we obtain

$$\begin{aligned} D_\infty^2 &= \frac{b^2}{2B_0^2}\frac{\int A(k_\perp)d\mathbf{k}_\perp}{\int k_\perp^2 A(k_\perp) d\mathbf{k}_\perp} = \frac{b^2}{2B_0^2}\frac{\langle a^2\rangle}{b^2} = \frac{b^2}{B_0^2}\frac{\tilde{\lambda}^2}{2} \\ D_\infty &= \frac{b}{B_0}\frac{\tilde{\lambda}}{\sqrt{2}} \quad (\text{high}\, R), \end{aligned} \tag{31}$$

which is the same as the previous result for 2D fluctuations,[22] where $\tilde{\lambda}$ is the ultrascale, defined[32] by $\tilde{\lambda}^2 \equiv \langle a^2\rangle/b^2$. Note that the ultrascale is the $k_\perp^{-2}$ moment of the total magnetic power spectrum. This type of field line diffusion is called Bohm diffusion because $D_\infty \propto b/B_0$. Similarly, the dimensionless diffusion coefficient becomes

$$D'_\infty = \frac{\sqrt{2}}{\pi}\frac{\tilde{\lambda}}{\ell_\perp} R \quad (\text{high}\, R) \tag{32}$$

so that $D'_\infty \propto R$. These limiting cases serve as a useful check that our application of the DD approach to noisy RMHD turbulence agrees with previous results of quasilinear diffusion in the quasi-1D (slab) limit at low R and Bohm diffusion in the quasi-2D limit at high R.

B. Random ballistic decorrelation

Now, we will change the third key assumption above, which specified σ^2 as a function of z based on DD, to instead assume RBD.[23] This treats the field line trajectories as ballistic, with each trajectory in a random direction, so that

$$\sigma^2 = \frac{\langle b_x^2\rangle}{B_0^2} z^2 = \frac{1}{2}\frac{b^2}{B_0^2} z^2. \tag{33}$$

This is based on the expectation that the TGK integral is dominated by low z values where the Lagrangian correlation remains strong, and that field line trajectories remain ballistic over that range because they have not decorrelated yet.

Using this RBD assumption, Eq. (21) becomes

$$\begin{aligned} D_\infty = \frac{1}{2B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k}) \\ \times \left[\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}k_\perp^2 \frac{\langle b_x^2\rangle}{B_0^2} z^2 - ik_z z\right) dz\right] d\mathbf{k}. \end{aligned} \tag{34}$$

Note that we have replaced the z-integral from 0 to ∞ with half the integral from $-\infty$ to ∞. This is valid because the complete integrand is invariant under the simultaneous transformation $z \to -z$ and $k_z \to -k_z$, which yields the same multiple integral with the z-integral from $-\infty$ to 0. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}k_\perp^2 \frac{\langle b_x^2\rangle}{B_0^2} z^2 - ik_z z\right) dz \\ = \frac{1}{k_\perp}\sqrt{\frac{2\pi B_0^2}{\langle b_x^2\rangle}} \exp\left(-\frac{k_z^2}{2k_\perp^2}\frac{B_0^2}{\langle b_x^2\rangle}\right) \end{aligned} \tag{35}$$

and

$$D_\infty = \frac{\sqrt{\pi}}{bB_0}\frac{1}{(2\pi)^{3/2}}\int \frac{P_{xx}(\mathbf{k})}{k_\perp}\exp\left(-\frac{k_z^2}{k_\perp^2}\frac{B_0^2}{b^2}\right) d\mathbf{k}. \tag{36}$$

Using Eq. (7) for a noisy RMHD field yields

$$D_\infty = \frac{\sqrt{\pi}}{bB_0}\frac{1}{2\pi}\int \frac{k_y^2 A(k_\perp)}{k_\perp}\frac{1}{K}\left[\int_0^K \exp\left(-\frac{k_z^2}{k_\perp^2}\frac{B_0^2}{b^2}\right) dk_z\right] d\mathbf{k}_\perp. \tag{37}$$

The result of the k_z-integration is an error function

$$\int_0^K \exp\left(-\frac{k_z^2}{k_\perp^2}\frac{B_0^2}{b^2}\right) dk_z = \frac{\sqrt{\pi}}{2} k_\perp \frac{b}{B_0} \mathrm{erf}\left(\frac{K}{k_\perp}\frac{B_0}{b}\right), \tag{38}$$

and after further manipulation we obtain

$$D_\infty = \frac{\pi}{4K}\frac{b^2}{B_0^2}\frac{\int k_\perp^2 A(k_\perp)\mathrm{erf}[(K/k_\perp)(B_0/b)] d\mathbf{k}_\perp}{\int k_\perp^2 A(k_\perp) d\mathbf{k}_\perp}. \tag{39}$$

In terms of dimensionless variables, we have

$$D'_\infty = \frac{R^2}{\pi}\frac{\int k'^2_\perp A(k'_\perp)\mathrm{erf}(\pi/2Rk'_\perp) d\mathbf{k}'_\perp}{\int k'^2_\perp A(k'_\perp) d\mathbf{k}'_\perp}. \tag{40}$$

As in the DD case, the expressions for D_∞ and D'_∞ from this RBD theory also simplify in the limit of very low or very high Kubo number. For $R \ll 1$, the argument of the error function is much greater than 1 for $k_\perp$ in the energy-containing range ($k_\perp \ell_\perp \lesssim 1$) and we can replace the error function with its asymptotic value of 1. Then, we again obtain the classic quasilinear limit

$$D_\infty = \frac{1}{2}\frac{b^2}{B_0^2}\ell_c, \quad D'_\infty = \frac{R^2}{\pi} \qquad (\text{low}\, R). \tag{41}$$

In the opposite limit of $R \gg 1$, the argument of the error function is much less than 1 for most of the energy-containing range. For $x \ll 1$, $\text{erf}(x) \approx (2/\sqrt{\pi})x$, and we obtain

$$D_\infty = \frac{\sqrt{\pi}}{2}\frac{b}{B_0}\ell_\perp, \quad D'_\infty = \frac{R}{\sqrt{\pi}} \qquad (\text{high}\, R), \tag{42}$$

where we have made use of Eq. (9). Since $\ell_\perp = \lambda_{c2}$, this expression for the diffusion coefficient is the same as that derived previously[23] (with the identification $b = b^{2D}$ because the present work uses only one fluctuation component, which in the high-R, quasi-2D limit functions like $\mathbf{b}^{2D}$). Thus, this RBD theory also tends toward Bohm diffusion in the high-R limit, and we again find a transition from the classic result for quasilinear diffusion in the quasi-1D (slab) limit at low R to the previous Bohm diffusion result in the quasi-2D limit at high R.

C. Numerical evaluation of analytic theories

In Figure 1, we present results for D_∞ vs. R based on numerical evaluation of the analytic DD and RBD theories, as well as the solution to an ordinary differential equation (ODE) model to be presented in Sec. V. To perform a numerical evaluation of analytic results, we must specify a power spectrum for the noisy RMHD turbulence. Here, we use

$$A(k_\perp) \propto \frac{1}{[1 + (k_\perp \lambda_\perp)^2]^{7/3}}, \tag{43}$$

though we stress that the analytic theories are not restricted to this choice. The above form for $A(k_\perp)$ has the properties that the omnidirectional 2D power spectrum at high $k_\perp$ is proportional to $k_\perp^{-5/3}$, representing Kolmogorov scaling in the perpendicular wavevectors in the inertial range,[32] and the low-$k_\perp$ behavior satisfies the requirements of strict homogeneity.[31]

We report all our numerical results for x- and y-distances in units of the perpendicular correlation scale $\ell_\perp$ and z-distances in units of K^{-1}. Thus, the numerical values we report are the values of the dimensionless (primed) variables. Evaluation of $\ell_\perp$ given $A(k_\perp)$ as specified above leads to the relation $\lambda_\perp = 2.678\ell_\perp$, and we also find that $\tilde{\lambda} = 1.546\ell_\perp$. We numerically evaluated the analytic results with the help of the Mathematica program (Wolfram Research, Inc.). In addition to D_∞, we also report a logarithmic derivative

$$\gamma \equiv \frac{R}{D_\infty}\frac{dD_\infty}{dR}, \tag{44}$$

which is a local power-law index such that $D_\infty \propto R^\gamma$. Figure 2 shows γ as a function of R.

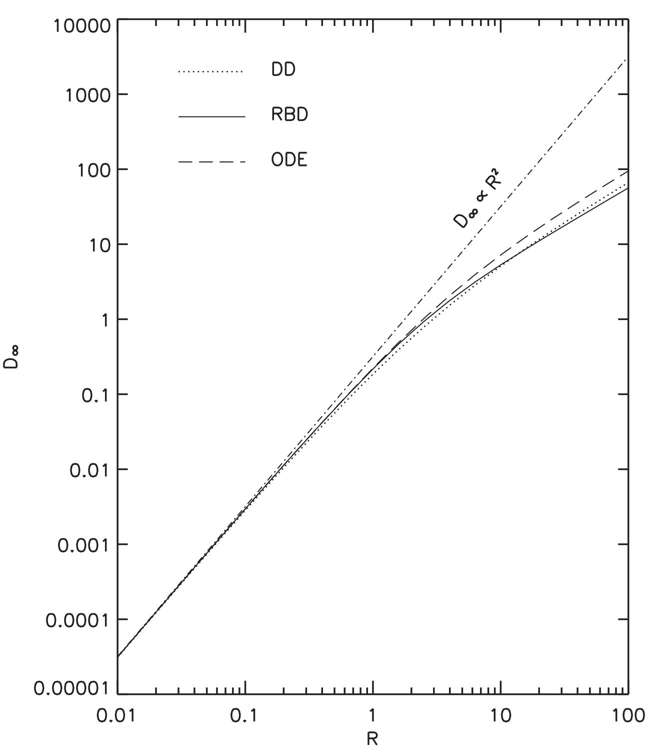

FIG. 1. Asymptotic magnetic field line diffusion coefficient D_∞ vs. the Kubo number R for noisy RMHD turbulence. Units are explained in the text. For each analytic theory (DD, RBD, or ODE), the behavior undergoes a transition from $D_\infty \propto R^2$ (for quasilinear diffusion) at $R \ll 1$ toward $D_\infty \propto R$ (for Bohm diffusion) at $R \gg 1$. For $R \lesssim 5$, which includes the range of Kubo numbers over which RMHD is likely to be applicable, we recommend the RBD theory for its ease of use and because it is corroborated by the more complete ODE theory.

V. EVOLUTION OF THE FIELD LINE RANDOM WALK

In addition to calculating the asymptotic diffusion coefficient of magnetic field lines, it is also interesting and useful to consider the evolution of the FLRW[33] as a function of the parallel distance z. We can define a running diffusion coefficient as

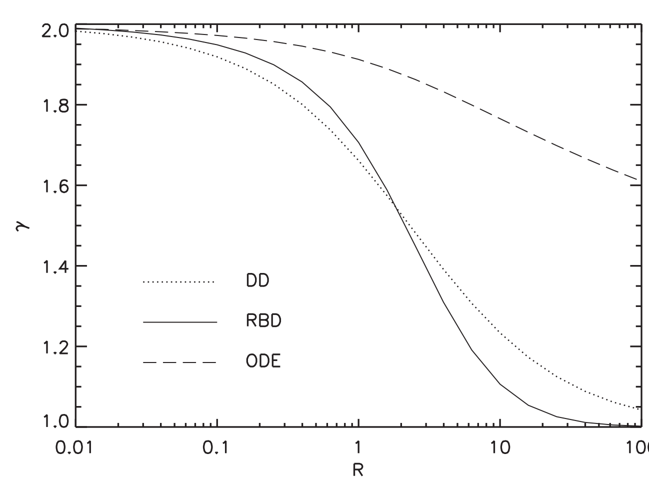

FIG. 2. Local power-law index $\gamma \equiv (R/D_\infty)(dD_\infty/dR)$ of the asymptotic magnetic field line diffusion coefficient D_∞ vs. the Kubo number R. For each theory (DD, RBD, or ODE), the index tends from 2 (for quasilinear diffusion) at low R toward 1 (for Bohm diffusion) at high R. Note that all three theories predict rather similar asymptotic diffusion coefficients (see Figure 1), so here we see that $\gamma(R)$ is quite sensitive to minor details in the functional dependence of $D_\infty(R)$.

$$D(z) \equiv \frac{1}{2}\frac{dV}{dz}. \tag{45}$$

Then, we describe the FLRW evolution in terms of the dependence of D on z. In the limit of large z, for cases of physical interest D typically approaches a constant asymptotic value, D_∞. Note that Eq. (13) can also be expressed as a second-order ODE

$$\frac{d^2V}{dz^2} = \frac{2}{B_0^2}\langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle, \tag{46}$$

or as an equivalent system of two first-order ODEs,

$$\frac{dV}{dz} = 2D, \tag{47}$$

$$\frac{dD}{dz} = \frac{1}{B_0^2}\langle b_x(0,0)b_x[\mathbf{x}_\perp(z),z]\rangle, \tag{48}$$

with the initial conditions $V(0) = D(0) = 0$.

Again making use of the assumptions of Corrsin's hypothesis and a Gaussian displacement distribution, the second equation becomes

$$\frac{dD}{dz} = \frac{1}{B_0^2}\frac{1}{(2\pi)^{3/2}}\int P_{xx}(\mathbf{k})e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}e^{-ik_z z}d\mathbf{k}. \tag{49}$$

Substituting the power spectrum for noisy RMHD turbulence, and performing the k_z-integration, we obtain

$$\frac{dD}{dz} = \frac{b^2}{2B_0^2}\frac{\int k_\perp^2 A(k_\perp)e^{-\frac{1}{2}k_\perp^2\sigma^2(z)}d\mathbf{k}_\perp}{\int k_\perp^2 A(k_\perp)d\mathbf{k}_\perp}\frac{\sin Kz}{Kz}, \tag{50}$$

or in terms of dimensionless variables,

$$\frac{dV'}{dz'} = 2D', \tag{51}$$

$$\frac{dD'}{dz'} = \frac{2}{\pi^2}R^2\frac{\int k_\perp'^2 A(k_\perp')e^{-\frac{1}{2}k_\perp'^2\sigma'^2(z')}d\mathbf{k}_\perp'}{\int k_\perp'^2 A(k_\perp')d\mathbf{k}_\perp'}\frac{\sin z'}{z'}. \tag{52}$$

In fact, part of the motivation for defining the dimensionless scaling $z' = Kz$ is that it simplifies the last factor in this equation.

To solve the differential equations, we need a closure to express σ^2 in terms of z. The DD and RBD theories correspond to integrable closures in which the right hand side of Eq. (50), or alternatively Eq. (52), is independent of V and D, and can be directly integrated to yield $D(z)$.[33] The DD theory for FLRW evolution uses $\sigma^2 = 2D_\infty z$ or $\sigma'^2 = 2D'_\infty z'$, which overestimates the actual variance at low z and is accurate at high z. For example, we can first solve the implicit equation for D_∞ as derived in Sec. IV A, and then substitute $\sigma^2 = 2D_\infty z$ into Eq. (50) and integrate to obtain $D(z)$. The RBD theory uses $\sigma^2 = (1/2)(b^2/B_0^2)z^2$, which can be substituted into Eq. (50), then integrating to obtain $D(z)$. The dimensionless form is $\sigma'^2 = (2/\pi^2)R^2 z'^2$. The RBD expression for σ^2 is accurate at low z and overestimates the actual variance at high z.

Another type of closure comes from identifying σ^2 with V in the ODE,[14,34,35] or by similar derivations.[36,37] We refer to this as self-closure of the ODE because there is no external input or specification. From now on, we will refer to this 2nd-order ODE with self-closure as the "ODE" theory. In the ODE theory, there is a continuous transition of the Lagrangian correlation [e.g., the right hand side of Eq. (50)] from the RBD expression at low z, where the RBD assumption of random ballistic trajectories is appropriate, to the DD expression at high z, where the DD assumption of asymptotic diffusion is appropriate. Thus, in principle, it is more accurate than the RBD or DD theory, but it has the disadvantage that deriving D_∞ generally requires a numerical solution of the ODE, whereas the RBD and DD theories provide analytic formulas for D_∞ as described in Sec. IV.

For the specific case described in Sec. IV C, we have numerically solved the ODE to determine D_∞ vs. R, as shown in Figure 1, and the power-law index γ vs. R, shown in Figure 2. The FLRW evolution is shown in Figure 3 for the DD, RBD, and ODE theories, based on numerical solutions to Eq. (52) for the corresponding specification of $\sigma^2(z)$. The results will be discussed in the Sec. VI.

Note that in the limits of low or high R, we can obtain analytic results for D_∞ from the ODE with self-closure. Considering Eq. (50), the right-hand side (the Lagrangian correlation) involves a product of two terms, a power-weighted average of $e^{-\frac{1}{2}k_\perp^2\sigma^2}$ and the oscillatory decay term $\sin(Kz)/(Kz)$. For $R \ll 1$, the oscillatory term decays faster, while the exponential remains close to 1. Therefore, in this limit σ^2 plays no role, and all three theories tend to the same limiting form. Approximating the exponential as 1, then even for the ODE theory, Eq. (50) decouples from equation Eq. (47) and we can simply integrate the former equation to obtain

$$D_\infty = \frac{b^2}{2B_0^2}\int_0^\infty \frac{\sin Kz}{Kz} = \frac{1}{2}\frac{b^2}{B_0^2}\frac{\pi}{2K} = \frac{1}{2}\frac{b^2}{B_0^2}\ell_c, \quad (\text{low } R) \tag{53}$$

which is the classic quasilinear result.

In the opposite limit of $R \gg 1$, we can instead assume that the averaged-exponential term decays much faster than the oscillatory term, i.e., over a z-distance much less than K^{-1}, so that we can approximate $\sin(Kz)/(Kz)$ as 1. Thus, in this limit, the ODE with self-closure is

$$\frac{d^2V}{dz^2} = \frac{1}{B_0^2}\frac{1}{2\pi}\int k_\perp^2 A(k_\perp)e^{-\frac{1}{2}k_\perp^2 V(z)}d\mathbf{k}_\perp \quad (\text{high } R). \tag{54}$$

Then, multiply both sides by dV/dz, construct total z-derivatives (which would not be possible if the oscillatory term remained), and integrate over z to obtain

$$\frac{1}{2}\left(\frac{dV}{dz}\right)^2 = C + \frac{1}{B_0^2}\frac{1}{2\pi}\int\left(\frac{-2}{k_\perp^2}\right)k_\perp^2 A(k_\perp)e^{-\frac{1}{2}k_\perp^2 V(z)}d\mathbf{k}_\perp \quad (\text{high } R), \tag{55}$$

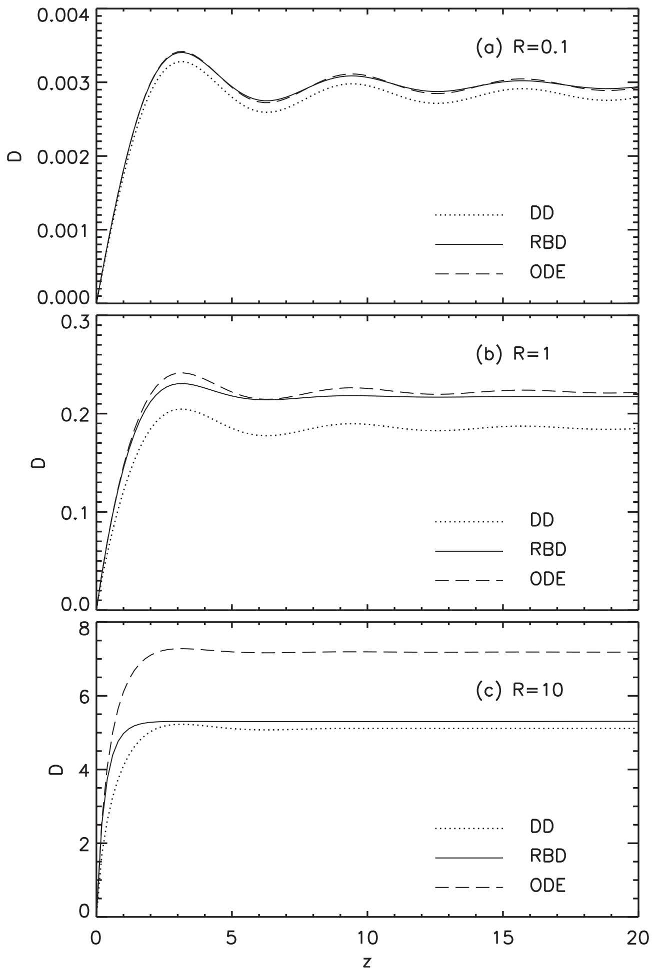

FIG. 3. Evolution of the field line diffusion coefficient D vs. the parallel displacement z for (a) $R = 0.1$, (b) $R = 1$, and (c) $R = 10$. Units are explained in the text. The more general ODE theory confirms results of the RBD theory for $R \lesssim 5$, both in terms of the evolution with z and the asymptotic diffusion coefficient (see Figure 1). Note that the ODE result varies with z like the RBD model (for random ballistic field line trajectories) at low z and like the DD model (for asymptotic field line diffusion) at high z, behavior which is most clearly seen for $R = 1$.

which is the first integral of the ODE,[14] for a constant of integration C. Simplifying this, and taking differences between $z = \infty$ and $z = 0$ [recalling the initial condition $D(0) = 0$ and assuming that $V(\infty) = \infty$], we obtain

$$\begin{aligned} 2D_\infty^2 &= \frac{2}{B_0^2}\frac{1}{2\pi}\int A(k_\perp)d\mathbf{k}_\perp = 2\frac{b^2}{B_0^2}\tilde{\lambda}^2 \\ D_\infty &= \frac{b}{B_0}\tilde{\lambda} \quad (\text{high}\,R), \end{aligned} \tag{56}$$

which is a factor of $\sqrt{2}$ higher than the DD result for high R [Eq. (31)] and also differs from the RBD result [Eq. (42)].

We can also use the first integral, which applies to the high-R limit, to examine the variation of $D(z)$ as $z \to \infty$. It can be seen that $C = 2D_\infty^2$, and note that for $Kz \gg 1/R$, the exponential is non-negligible only for $k_\perp \ell_\perp \ll 1$. There $A(k_\perp)$ can be approximated as a constant, $A(0)$, assuming that A is nearly constant in the energy-containing range of the fluctuations. Then after some manipulations, it can be shown that for $R \gg 1$ and $Kz \gg 1/R$, the ODE gives $D(z) = D_\infty - O(z^{-1})$.

VI. DISCUSSION

In this work, we have developed analytic theories of the magnetic FLRW in noisy RMHD turbulence. We have considered the FLRW in terms of both the asymptotic diffusion coefficient D_∞ and the evolution of D as a function of z. These theories are based on Corrsin's hypothesis and can model quasilinear diffusion and Bohm diffusion, but do not account for some trapping effects. According to previous work, the effects of percolation or trapping are important for nearly 2D fluctuations[3,15] or a high Kubo number R.

A direct comparison can be given from our results on several assumptions regarding the dominant factor that produces decorrelation. From Figure 1, we can compare the results of DD, RBD, and ODE theories for D_∞ as a function of R. All of these assume Corrsin's hypothesis and a Gaussian displacement distribution, but they differ in their assumptions for the variance of the perpendicular displacement distribution as a function of the displacement z parallel to the large-scale field. The DD theory employs the variance of asymptotic diffusion, and can, therefore, be thought to be most appropriate for high z. RBD uses the variance of random ballistic trajectories, and accordingly is most appropriate for low z. The ODE formulation uses a variance determined self-consistently, which evolves from RBD to DD behavior as a function of z.

The transition from RBD to DD behavior can be directly visualized in Figure 3. In all cases, the ODE result for D initially varies with z like the RBD model (for random ballistic field line trajectories) and at high z it varies like the DD model (for asymptotic field line diffusion). This behavior is most clearly seen for $R = 1$, where the RBD model ceases to oscillate after $z \approx 5$ because it overestimates the variance of $P(\mathbf{x}_\perp|z)$ and its expression for the Lagrangian correlation declines too quickly, so that D ceases to evolve. The DD model is more accurate at high z, and indeed the ODE result varies in the same way as DD for large z. Nevertheless, the Lagrangian correlation is already quite low by that high-z range, and the evolution of the diffusion coefficient $D(z)$ is dominated by the low-z region where the Lagrangian correlation is still substantial. Therefore, the RBD result is quite close to the ODE result at all z and the DD result is substantially lower. The match between the ODE and RBD results is even closer at R $= 0.1$. However, by $R = 10$ we see that the ODE results are quite different from both the RBD and DD results. Evidently, the z-range over which the Lagrangian correlation remains significant in the ODE model is now greater than the z range over which the field line trajectories remain ballistic but not great enough to accurately model the displacement distribution as diffusive.

All the theories undergo a transition from quasilinear diffusion ($D_\infty \propto R^2$) at $R \ll 1$ to Bohm diffusion ($D_\infty \propto R$) at $R \gg 1$. In terms of γ, a logarithmic derivative of D_∞ with respect to R, for which $D_\infty \propto R^\gamma$ locally, the results undergo a transition from $\gamma = 2$ at $R \ll 1$ toward $\gamma = 1$ as $R \to \infty$ (see Figure 2).

For $R \ll 1$, the quasilinear limit, all the theories agree for reasons discussed above in Sec. V. For higher R, the DD and RBD theories differ by at most 23%, which occurs in the limit $R \to \infty$. It is interesting that there is a crossover, with $D_{\rm RBD} > D_{\rm DD}$ for $R \le 13$ and $D_{\rm RBD} < D_{\rm DD}$ for $R > 13$. At low R, the ODE result is very close to the RBD result. It begins to deviate noticeably at $R \sim 1$. Thereafter, the difference increases with increasing R, and is 20% at $R \approx 5$. With such confirmation from the ODE result, we conclude that the RBD theory is slightly superior to the DD theory.

The three theories differ substantially in their ease of use: RBD provides an explicit formula, while DD gives an implicit formula, and ODE requires solving a differential equation. Based on a balance of accuracy and ease of use, we prefer the use of RBD for values of R up to 5. We also note that replacing the DD assumption with RBD in the nonlinear guiding center (NLGC) theory[38] of the perpendicular diffusion of energetic charged particles has led to a substantially improved agreement between NLGC theory and direct simulation results for 2D+slab turbulence.[39]

According to our experience in the study of 2D+slab turbulence,[33] when the ODE results agreed with the DD or RBD results, those models were in reasonable agreement with direct computer simulation results. Conversely, when the ODE results differed substantially from DD and RBD results, all results, even for the ODE, were in substantial disagreement with computer simulations. Therefore, we recommend that the models presented in this work be considered to apply for $R < 5$. Note that indeed, a transverse magnetic fluctuation model such as the noisy RMHD model used here would not be expected to apply for physical situations with $R \gg 1$.

Presumably the disagreement among the results from these models is due to trapping and/or percolation effects, which have been stated to be inconsistent with Corrsin's hypothesis. For example, for a mathematically analogous problem of convective transport in 2D incompressible fluid flow, the regime of high Kubo number corresponds low fluctuation frequency ω, and it has been argued[15] that Bohm diffusion results[13] do not apply in that limit, and that percolative scaling is more appropriate. Similar arguments with regard to the FLRW stated more strongly and specifically that Corrsin's hypothesis leads to Bohm diffusion in the limit of large R, and for $R > 1$ Corrsin's hypothesis is not valid.[40,41] Note, however, that even when Corrsin's independence hypothesis *per se* is violated for a range of z values (e.g., at low z), it is not clear that theories that employ the hypothesis will necessarily yield inaccurate values of the diffusion coefficient, e.g., when applied over a wider z range.[23] The present work suggests that the theories presented here should remain valid up to $R \sim 5$, and further work could use computer simulations to verify their range of accuracy.

Regarding the variation of the resulting diffusion with model parameters, we note that the 2D+slab system admits two parameters, whereas the noisy RMHD system has the Kubo number R as its only parameter. For slab+2D models, one can observe clear regimes of quasilinear and Bohm diffusion and trapping behavior. For the noisy RMHD system, behavior ranging between quasilinear and Bohm regimes (related to γ; see Figure 2) is characterized by changes vs. R. We find that γ decreases over the range of applicability of our theories, which is consistent with recent simulation results for isotropic turbulence that is stretched in **k**-space.[29]

Note that all three theories predict rather similar asymptotic diffusion coefficients (see Figure 1), but their results for the power-law index γ are quite different. We conclude that $\gamma(R)$ is quite sensitive to minor details in the functional dependence of $D_\infty(R)$. We, therefore, caution against detailed interpretation of $\gamma(R)$, and for applications it is more relevant to consider $D_\infty(R)$. In any case, it is clear from Figure 1 that for $R \gtrsim 1$ the quasilinear expression extrapolated from low R has a large disagreement with the more detailed analytic theories presented in this work. We propose that an explanation of the FLRW requires another type of diffusion, which we consider to be Bohm diffusion, combined with quasilinear diffusion.

The problem of the FLRW in RMHD has proven to be equally as rich as the slab+2D model in its range of allowed behavior. The RMHD model, like the 2D+slab model, is a three dimensional model, with no ignorable coordinate. Furthermore, it is closely related to models that are widely employed in solar physics and astrophysics, so that the results for the homogeneous RMHD FLRW may have broad application. Notably, the problem of field line separation in weakly three dimensional transverse turbulence, as well as several other problems related to magnetic connectivity,[42] may be addressed conceptually and analytically, based on the FLRW results derived here. Finally, we note that RMHD is a dynamical model, and therefore the results derived here for the magnetostatic noisy RMHD field can provide a useful baseline for further exploration of dynamical, anisotropic MHD turbulence.

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