The Astrophysical Journal, 633:L49–L52, 2005 November 1

SUPPRESSED DIFFUSIVE ESCAPE OF TOPOLOGICALLY TRAPPED MAGNETIC FIELD LINES

P. Chuychai,[1,2,3] D. Ruffolo,[3] W. H. Matthaeus,[2] and G. Rowlands[4]

Received 2005 August 10; accepted 2005 September 19; published 2005 October 12

ABSTRACT

Many processes in astrophysical plasmas are directly related to magnetic connection in the presence of turbulent fluctuations. Even statistically homogeneous turbulence can contain closed topological structures that inhibit otherwise random transport of field line trajectories, thus temporarily trapping certain trajectories. When a coherent random field perturbation is added, the trapped field lines can escape diffusively but at a suppressed rate that is much lower than what would be estimated based on the perturbation field alone. Here we demonstrate both trapping and escape, and show, using a novel quasi-linear theory, how to compute the suppressed diffusion that affects the escape from the trapping structure. The effect is relevant to understanding filamentary magnetic connection in interplanetary space and the observed dropouts in moderately energetic particles from impulsive solar flares. Expressed here in terms of a magnetic field line random walk, this phenomenon also has analogies in a broad range of dynamical systems that evolve as an incompressible flow in phase space with a coherent perturbation.

Subject headings: diffusion — magnetic fields — turbulence

1. INTRODUCTION

The behavior of an ensemble of magnetic field lines subject to transverse fluctuations is in direct analogy to phase-space trajectories of dynamical systems that obey Liouville's theorem. Therefore, the transport of magnetic field lines having a random perturbation is a model for certain volume-preserving mappings in nonlinear dynamics. Furthermore, transport of field lines, closely related to transport of charged particles (Jokipii 1966), is of fundamental importance in space and astrophysics and has a great impact on heat conduction (Chandran & Cowley 1998), cosmic-ray transport (Jokipii & Parker 1968), and magnetic field complexity (Matthaeus et al. 1995). Recently, we showed how puzzling observations of persistent sharp gradients of observed solar energetic particle (SEP) intensities might be explained by topological trapping of field lines by closed quasi–two-dimensional magnetic islands that inhibit field line transport, and therefore particle transport (Ruffolo et al. 2003; see also Giacalone et al. 2000; Zimbardo et al. 2004), despite the presence of a random field perturbation in the solar wind ("slab" turbulence) that is coherent over the two-dimensional islands. Here we examine the phenomenon of field line trapping and escape, and show, using a novel quasi-linear theory, that escape occurs not at the expected rate but at a suppressed diffusive rate. The suppression is due to interference between the trapping two-dimensional field and the escape-producing slab field. The net diffusion rate is low until the field line leaves the trapping zone and the normal rate is recovered. The result is an extended filament of magnetic connection that is unusually resistant to the slab perturbations.

[1] Department of Physics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand.
[2] Bartol Research Institute, University of Delaware, Newark, DE 19716; paeng@bartol.udel.edu, yswhm@bartol.udel.edu.
[3] Department of Physics, Faculty of Science, Mahidol University, Rama VI Road, Bangkok 10400, Thailand; david_ruffolo@yahoo.com.
[4] Department of Physics, University of Warwick, Coventry CV4 7AL, UK; g.rowlands@warwick.ac.uk.

2. PHENOMENOLOGY

The magnetic field line is defined to be tangent everywhere to the magnetic field $\boldsymbol{B}$. If $d\boldsymbol{l}$ is an arc length, the lines of force are defined by the differential equation

$$d\boldsymbol{l} \times \boldsymbol{B} = 0. \quad (1)$$

The possibly stochastic character of an ensemble of solutions (field lines) obtained from this equation depends crucially on the spatial complexity of the magnetic field throughout the region of interest. In this Letter, we use a simple model to study the topological inhibition of the random walk of magnetic field lines. The total magnetic field can be written as $\boldsymbol{B}(x, y, z) = B_0\hat{\boldsymbol{z}} + \boldsymbol{b}(x, y, z)$, where $B_0\hat{\boldsymbol{z}}$ is the mean field and $\boldsymbol{b}$ is the fluctuation perpendicular to the mean field. The fluctuation is the sum of a two-dimensional field and slab turbulence, which we write in the form

$$\boldsymbol{b}(x, y, z) = \boldsymbol{b}^{2D}(x, y) + \boldsymbol{b}^{slab}(z). \quad (2)$$

In general, we can write $\boldsymbol{b}^{2D}(x, y) = \nabla \times a(x, y)\hat{\boldsymbol{z}}$, where $a(x, y)$ is called the potential function. For the pure two-dimensional case, the field lines must follow level surfaces (contours) of $a(x, y)$. Fluctuations in the solar wind are found to be well described by such a superposition of slab and two-dimensional fluctuations (Matthaeus et al. 1990; Bieber et al. 1994, 1996). Substituting time t for distance z, this model also applies to physical or industrial processes with a systematic two-dimensional flow (the velocity field is analogous to $\boldsymbol{b}^{2D}$) and random, time-dependent shaking by an external force (the slab fluctuation).

For a model of a single island of topological trapping in a two-dimensional field, here we set $a(x, y)$ as a Gaussian function:

$$a(x, y) = A \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) = A \exp\left(-\frac{r^2}{2\sigma^2}\right), \quad (3)$$

where A is the maximum value at the center of the Gaussian and σ represents the width of the Gaussian. Without the slab

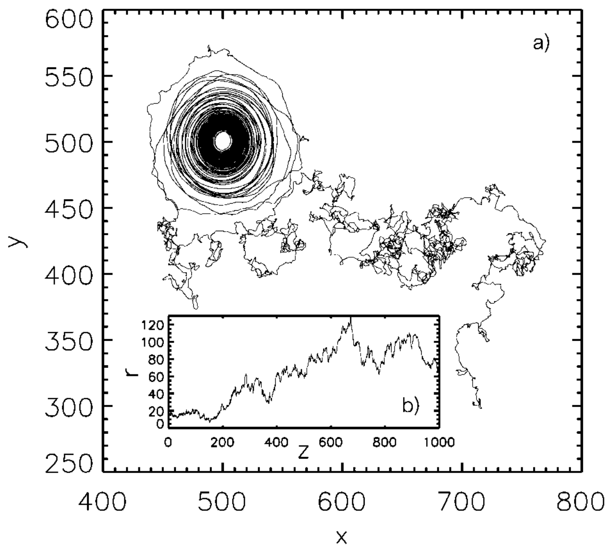

FIG. 1.—(*a*) Orbit in the *x*-*y* plane of a selected field line that lies deep in the trapping island, showing the predominantly cyclic trajectory, which gradually is transported out of the trapping region, where the motion becomes highly irregular, and is an almost unconstrained random walk. (*b*) Plot of the radial coordinate $r = (x^2 + y^2)^{1/2}$ of the field line in (*a*) vs. the parallel coordinate. At short distances, the radial position remains near its initial value $r_0 = 15$; the trapping island is of width $\sigma = 30$. After around $z = 150$, the field line breaks out of the trapping structure and random walks with a much larger amplitude.

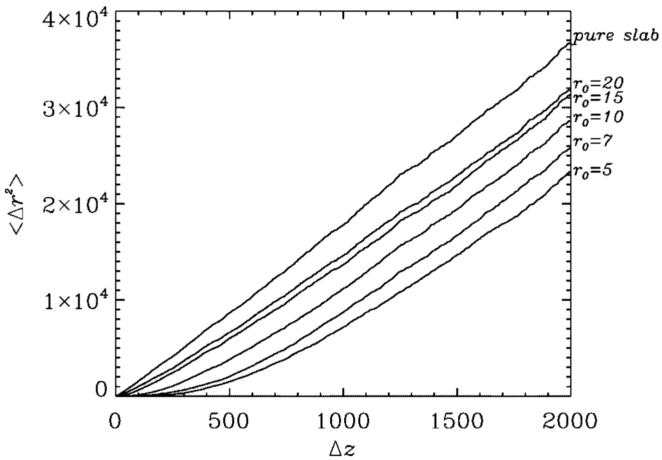

FIG. 2.—Displacement squared in radial component *r* of ensembles of field lines. Smaller initial *r* corresponds to greater depth in the two-dimensional structure ($\sigma = 10$) and therefore more potent trapping. All field lines eventually are transported diffusively with the full slab diffusive rate, but field lines starting more deeply in the trapping island experience an effective delay in attaining this rate.

field, the field line trajectory is a helical orbit along a cylinder of constant $a(x, y)$ with a constant angular "velocity" (in terms of the distance *z*) $K = a(r_0)/B_0\sigma^2 = [b^{2D}(r_0)/B_0]/r_0$, where r_0 is the starting radius. On the other hand, for pure slab turbulence, the field lines undergo a random walk with correlation length l_c.

We numerically explore how field lines behave under the combined influence of these two effects: Gaussian two-dimensional plus slab turbulent fields. We simulate the field lines starting at different radii of the Gaussian function $a(x, y)$ and examine the possible diffusive behavior of their transverse displacements. In order to obtain the field line trajectories, we solve the field line equations from equation (1):

$$\frac{dx}{dz} = \frac{b_x^{2D} + b_x^{\rm slab}}{B_0}, \quad \frac{dx}{dz} = \frac{b_y^{2D} + b_y^{\rm slab}}{B_0}. \qquad (4)$$

The magnetic fields are synthesized as follows. For the two-dimensional field, we directly calculate the magnetic field in real space from the two-dimensional potential function. For the slab turbulence, the field is generated in wavenumber space by specifying the shape of the magnetic spectrum and choosing random phases of the Fourier amplitudes. We choose the Kolmogorov spectrum,

$$P_{xx}(k_z) = P_{yy}(k_z) = \frac{C}{[1 + (k_z l_z)^2]^{5/6}}, \qquad (5)$$

where *C* is a normalization constant and l_z is a coherence length related to l_c. The spectrum is flat when $k_z \ll 1/l_z$ and rolls over at $k_{0z} = 1/l_z$. For $k_z \gg k_{0z}$, the spectral shape is proportional to $k^{-5/3}$. We use an inverse fast Fourier transform to convert the slab field into real space. Equation (4) is solved by a fourth-order Runge-Kutta method with adaptive time stepping regulated by a fifth-order error estimate (Press et al. 1992).

Figure 1*a* shows a typical trajectory of a field line in this model field, projected in the *x*-*y* plane ($\sigma = 30$, $r_0 = 15$). The field line is temporarily trapped in nearly circular orbits within the two-dimensional island. When it eventually leaves the two-dimensional island, the trajectory becomes irregular due to the slab turbulence. Figure 1*b* is a plot of radius $r = (x^2 + y^2)^{1/2}$ versus distance *z* of the field line shown in Figure 1*a*. The field line experiences small random changes in radius as it is dominantly influenced by the two-dimensional field. It becomes a large-scale random walk when the two-dimensional field is not dominant.

Since typical solutions like Figure 1 are irregular, we examine the statistics of many field lines. We trace 2500 field lines and measure the average squared radial displacement in the *x*-*y* plane, $\langle \Delta r^2 \rangle$, versus distance *z*. Note that the field lines are traced for only 10% of the length of the simulation box in the *z*-direction (to avoid periodicity effects inherent in our field generation method). In this way, our results differ fundamentally from the periodic-stochastic transition that occurs in periodic toroidal domains, a topic well studied in laboratory fusion and in nonlinear dynamics (Rosenbluth et al. 1966). The present results are a model for trapping and escape in an unbounded or homogeneous plasma, appropriate to space and astrophysical systems, or in the time domain, to physical or industrial processes of long duration.

The fields are generated in box sizes 1000 × 1000 × 100,000 in units of the parallel coherence scale. The grid sizes are $N_x = 4000$, $N_y = 4000$, and $N_z = 4{,}194{,}304$. We set $\sigma = 10$, $l_z = 1.0$, $(\delta b^{\rm slab}/B_0)^2 = 12.5$, and $(\delta b^{2D}/B_0)^2 = 12.5$, where $\delta b^{\rm slab}$ and δb^{2D} denote the rms of the slab and two-dimensional fields, respectively, averaged over the entire box. With these parameters, the two-dimensional field is very strong near the center of the two-dimensional island, as the size of the Gaussian width is very small compared to the *x*-*y* simulation region. Each field line starts at $r = r_0$ and a random azimuthal angle.

Figure 2 illustrates $\langle \Delta r^2 \rangle$ versus *z* for various initial radii in the two-dimensional+slab case and in the pure slab case. When

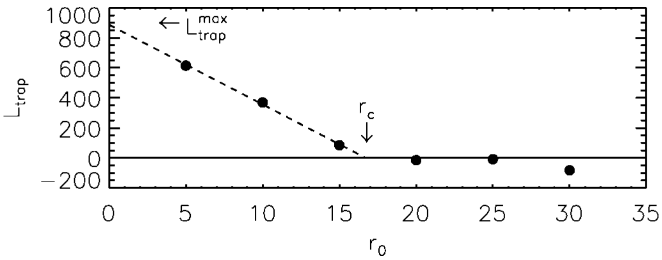

FIG. 3.—Illustration of the definitions of r_c and $L_{\rm trap}^{\rm max}$.

field lines start inside the two-dimensional island, the slopes of the mean square radial displacements change systematically and are initially much less than the slab rate. There is a delay in attaining faster diffusion rates, implying that field lines are trapped temporarily due to the strong two-dimensional field. Only at much greater distances do the field lines eventually attain strong diffusive transport at almost the full slab rate. If we trace the slope at long distance back to the x-axis, then we can define an effective trapping length ($L_{\rm trap}$), and we find that this scales systematically with various parameters.

When the field lines start more deeply inside the two-dimensional island, $L_{\rm trap}$ becomes longer. Furthermore, we can estimate the boundary of trapping (r_c) and the maximum of the trapping length for each case ($L_{\rm trap}^{\rm max}$) from the plot of $L_{\rm trap}$ versus starting radius r_0, as shown in Figure 3. When varying parameters such as the strength of two-dimensional and slab fluctuation, correlation length, and the width of the Gaussian, an empirical result is

$$L_{\rm trap}^{\rm max} \propto \left[\left(\frac{\delta b^{\rm 2D}}{B_0}\right)^2\right]^{0.62}\left[\left(\frac{B_0}{\delta b^{\rm slab}}\right)^2\right]^{0.74}\frac{l_z^{0.53}}{\sigma^{1.5}}. \quad (6)$$

On the other hand, the trapping boundary r_c depends only on the length scales of two-dimensional and slab fields and is independent of the strength of the fluctuations. We have carried out a numerical experiment in which field lines are started uniformly in a circle that has a size larger than σ. For a very strong two-dimensional field, a sharp trapping boundary appears. The field lines inside the trapping boundary are trapped while the field lines outside this boundary quickly diffuse away.

3. THEORY OF SUPPRESSED DIFFUSION

The simulations show that when two-dimensional and slab fields are superimposed, the field lines do not follow the contours of $a(x, y)$ but are also not fully diffusive with the slab rate. The field lines are trapped near the center of the Gaussian and rapidly diffuse with the slab rate only at a radial distance $r \gg \sigma$. If we consider the region where the two-dimensional field is much stronger than the slab component, we can treat the slab fluctuation as a perturbation and apply a quasi-linear approach. Specifically, we assume the orbit is unchanged by the slab field at leading order and therefore its transverse position (x, y) traces a circle advancing at angular velocity K with increasing z, that is, $x = r_0 \cos(Kz + \varphi)$, $y = r_0 \sin(Kz + \varphi)$. At the next order, the mean squared fluctuation in radius is

$$\langle \Delta r^2 \rangle = \frac{1}{B_0^2}\int_0^{\Delta z}\int_0^{\Delta z}\langle b_r(z')b_r(z'')\rangle dz'\, dz'', \quad (7)$$

where $b_r(z)$ is the projection of the slab field in the radial direction, which changes during the circular motion. In terms of the slab correlation function (Jokipii 1973), $R_{xx}(\Delta z')$, with $\Delta z' \equiv z'' - z'$,

$$\langle \Delta r^2 \rangle = \frac{1}{B_0^2}\int_0^{\Delta z}\int_{-\infty}^{\infty} R_{xx}(\Delta z')\cos(K\Delta z')d\Delta z'\, dz', \quad (8)$$

where the integration over all $\Delta z'$ is a valid approximation when $\Delta z \gg l_c$. In terms of the power spectrum P_{xx}, we have

$$D_{rr} = \frac{\langle(\Delta r)^2\rangle}{2\Delta z} = \sqrt{\frac{\pi}{2}}\frac{P_{xx}(K)}{B_0^2} = D_{\rm slab}\frac{P_{xx}(K)}{P_{xx}(0)}, \quad (9)$$

where $D_{\rm slab}$ is the standard (Jokipii 1966) quasi-linear slab result. The theoretical result in equation (9) tells us that the radial motion of the field lines deeply inside the two-dimensional island is diffusive and is associated with the slab power spectrum at the wavenumber resonant with the two-dimensional angular velocity at the original radius.

To confirm the theory, we compute $\langle \Delta r^2\rangle/(2\Delta z)$ from the simulations as shown in Figure 2 and compare this with the suppressed diffusion theory. Note that this theory is only expected to hold at low Δz, when field lines are still near $r = r_0$. The comparison is presented in Figure 4. The field lines starting well inside the Gaussian (of width $\sigma = 10$), such as $r_0 = 5$ and 7, give good agreement with the theory while the discrepancy between theory and simulations increases when we start the field lines away from the center of the Gaussian. At long distances, the field lines starting at different r_0 spread at the same rate, which is almost the rate of the pure slab case. The diffusion rate for all cases at large distance is $\sim 2D_{\rm slab}$ since $\langle \Delta r^2\rangle = \langle \Delta x^2\rangle + \langle \Delta y^2\rangle$ responds to the rapid field line random walk in each Cartesian coordinate (in the x-y plane), not only in the radial direction as when $r \approx r_0$. The diffusion rate in the two-dimensional+slab case is slightly lower than in the pure slab case at long distances because there is a small probability that escaped field lines reenter the two-dimensional island and are again trapped. In addition, the synthetic slab field lines are periodic in z, so there is a slightly higher probability that the field lines return to the trapping center. However, we expect

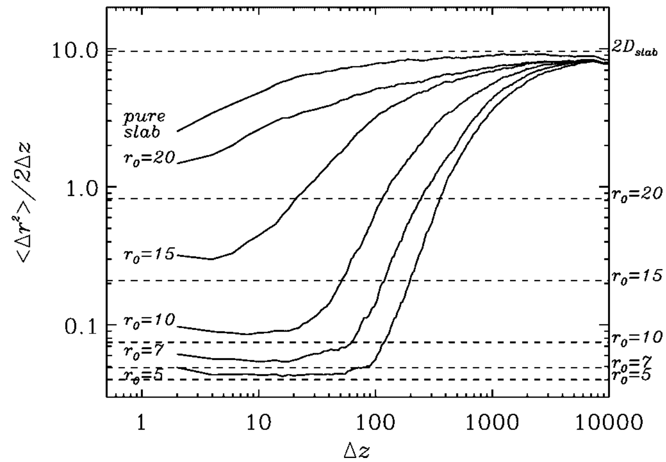

FIG. 4.—Suppressed and full slab diffusion rates from numerical simulations (*solid lines*) and the theory of suppressed diffusion for small z (*horizontal dashed lines*). The theory accurately describes the suppressed diffusion of a set of field lines that starts deep inside the two-dimensional structure (here $\sigma = 10$).

that if there were no periodicity in z, and the simulation could be continued to much larger z, the radial diffusion coefficient would converge precisely to the slab rate.

Our study shows that the two-dimensional field can temporarily trap field lines and suppress the field line random walk at short to intermediate distances. For large distances, all field lines escape the two-dimensional topology and diffuse asymptotically at the slab rate. For the field lines starting deeply inside the two-dimensional island, the suppressed diffusion arises because the rapid motion around the trapping island decorrelates the radial component of the perturbation. We can use quasi-linear theory to calculate the suppressed transport rate, which parametrically depends on the initial radius, a measure of the degree of trapping. This mechanism helps us to understand complicated systems such as the two-dimensional+slab turbulent magnetic field (Ruffolo et al. 2003), which is thought to be a reasonable model for the interplanetary magnetic field. In this model, the two-dimensional field is turbulent and there are many islands of irregular shape. When the field lines start within a certain region that is relevant to the injection region of SEPs, the field lines starting near local maxima or minima of the two-dimensional islands can be trapped within the islands while field lines starting between islands, or near the local saddle points, rapidly diffuse. Hence, the observed filamentation of field lines (Mazur et al. 2000) occurs at intermediate distances due to topological trapping, which is enhanced due to suppressed escape. At long distances, the field lines diffuse at the unsuppressed rate.

4. CONCLUSIONS

In conclusion, we have found that a strong two-dimensional field can inhibit the random walk of field lines due to a slab field component. The simulations show that when we start the field lines inside the two-dimensional island, the diffusion of field lines systematically changes with a delay at the beginning due to the strong two-dimensional field. The trapping boundary depends only on the topological scale σ of the two-dimensional island and the correlation scale l_c of the slab turbulence. The field lines located near the maximum of the two-dimensional potential function diffuse outward at a lower rate than when they are outside the two-dimensional island. We theoretically explain the suppression of the field line diffusion inside the two-dimensional island by a quasi-linear theory, which is confirmed by simulations. Finally, our study of the suppression of the random walk of field lines is applicable to any system that consists of a systematic flow in two dimensions on which is superimposed a spatially coherent random walk.

We acknowledge support by grants from the Thailand Research Fund, the Ratchadapisek Sompoj Fund of Chulalongkorn University, US National Science Foundation grant ATM-0105254, and NASA grant NAG5-11603. G. R. thanks Mahidol University for their kind hospitality and for travel support from the Thai Commission for Higher Education.

REFERENCES

Bieber, J. W., Matthaeus, W. H., Smith, C. W., Wanner, W., Kallenrode, M.-B., & Wibberenz, G. 1994, ApJ, 420, 294

Bieber, J. W., Wanner, W., & Matthaeus, W. H. 1996, J. Geophys. Res., 101, 2511

Chandran, B. D. G., & Cowley, S. C. 1998, Phys. Rev. Lett., 80, 3077

Giacalone, J., Jokipii, J. R., & Mazur, J. E. 2000, ApJ, 532, L75

Jokipii, J. R. 1966, ApJ, 146, 480

———. 1973, ARA&A, 11, 1

Jokipii, J. R., & Parker, E. N. 1968, Phys. Rev. Lett., 21, 44

Matthaeus, W. H., Goldstein, M. L., & Roberts, D. A. 1990, J. Geophys. Res., 95, 20673

Matthaeus, W. H., Gray, P. C., Pontius, D. H., Jr., & Bieber, J. W. 1995, Phys. Rev. Lett., 75, 2136

Mazur, J. E., Mason, G. M., Dwyer, J. R., Giacalone, J., Jokipii, J. R., & Stone, E. C. 2000, ApJ, 532, L79

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes in FORTRAN: The Art of Scientific Computing (Cambridge: Cambridge Univ. Press)

Rosenbluth, M. N., Sagdeev, R. Z., Taylor, J. B., & Zaslavski, G. M. 1966, Nucl. Fusion, 6, 297

Ruffolo, D., Matthaeus, W. H., & Chuychai, P. 2003, ApJ, 597, L169

Zimbardo, G., Pommois, P., & Veltri, P. 2004, J. Geophys. Res., 109, A02113